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State-of-the-art on the teaching and learning of mathematical modelling

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The world is currently facing many problems:

- Social problems such as economical inequality, limited access to water, food, … of huge parts of the world population, …..
- Environmental problems such as problems caused by earth quakes, volcanos, Tsunamis, flooding, fire …..

What can mathematics and specifically mathematical modelling contribute to prepare school students addressing these problems?

Structure

1. **Theoretical Perspectives** on mathematical modelling including the modelling process
2. **Modelling competencies** - definition, measurement and fostering: results of a systematic literature survey
3. Further results of **empirical studies**: cognitive, meta-cognitive and affective aspects; teacher interventions and adaptive teacher support; new media
4. **Looking ahead**

1. Theoretical perspectives on mathematical modelling

Central aim of mathematical modelling:

to answer the questions

- **why shall students in school learn mathematics** and
- **how** can we **enable students** to **use mathematics** in order to **solve real world problems** or more specifically to **use mathematics in real world situations in a critical way?**

Milestone for recent international initiatives for implementing modelling approaches in mathematics education:

Hans Freudenthal's Symposium "**How to teach mathematics so as to be useful**" in 1968, **ICME-1**, in Exeter, proceedings published as first volume of Educational Studies in Mathematics in 1969.

In his welcome speech, "**Why to teach mathematics so as to be useful**", Freudenthal made a strong plea to change mathematics education, to include real world examples and modelling into mathematics education in order to make mathematics **more meaningful** for **students**. The connection to real life, **mathematizing the world**, was important for him.

He criticises the usual teaching approaches, unfortunately still common in many classrooms until today:

"The huge majority of students are not able to apply their mathematical classroom experiences, neither in the physics or chemistry school laboratory nor in the most trivial situations of daily life."

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What does it mean to teach mathematics so as to be useful, to teach mathematics in such a way that students can use it in order to solve real world problems?

Currently accepted approach (not the only one) to answer these questions is the **modelling approach**:

What do we mean by mathematical modelling?

Using mathematics in order to solve real world problems with mathematical methods.

Learning modelling means: **develop students' competencies to use their mathematics for the solution of problems of their daily life and from sciences**

Relation between mathematics and reality is important, going back and forth.

Modelling is an activity, a process.

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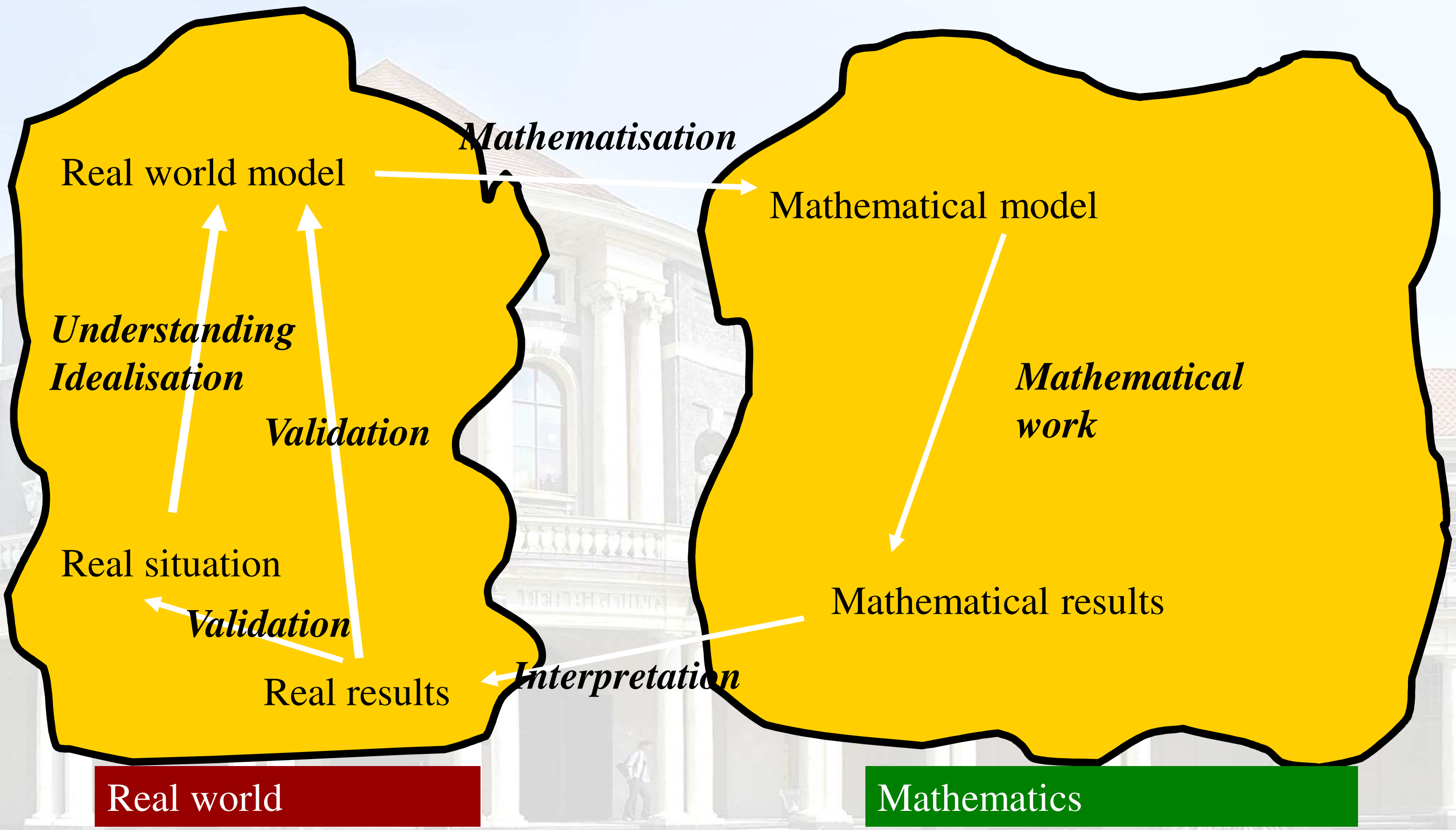
Classical modelling cycle



Simple modelling example

Lighthouse Problem

Traditionally, there existed many lighthouses all over the world, which were supposed to steer and guide the shipping traffic. Chile with its large and intricate coastline of 4000 km and three waterways between the Pacific Ocean and the Atlantic Ocean maintains 650 lighthouses from the border with Peru to the Atlantic ocean.
Their task is to mark dangerous coastlines, hazardous shoals, reefs, and safe entries to harbors.
On 18 September 1857, the first lighthouse in Chile, the "Faro Angeles", was inaugurated in Valparaíso. In 2009 18 were still inhabited.

Lighthouse Problem

In 1950 to 1951, the Lighthouse of la Serena was built at the coast of Chile. It is 25 metres high. It was supposed to warn ships of approaching Chile's coastline. the lighting system was removed afterwards, leaving it as notable point of reference in later charts and publications.

Many questions are possible, one could be: How far off the coastline was a ship when the crew was able to see the lighthouse for the first time ?

Copyright Alfonso Claps, Wikipedia

Original lighthouse problem

Lighthouse Roter Sand

In 1884, the 30.7 m high "Roter Sand" lighthouse was built directly on the coast in the Bay of Bremen. Its beacon was intended to warn ships that they were approaching the coast.

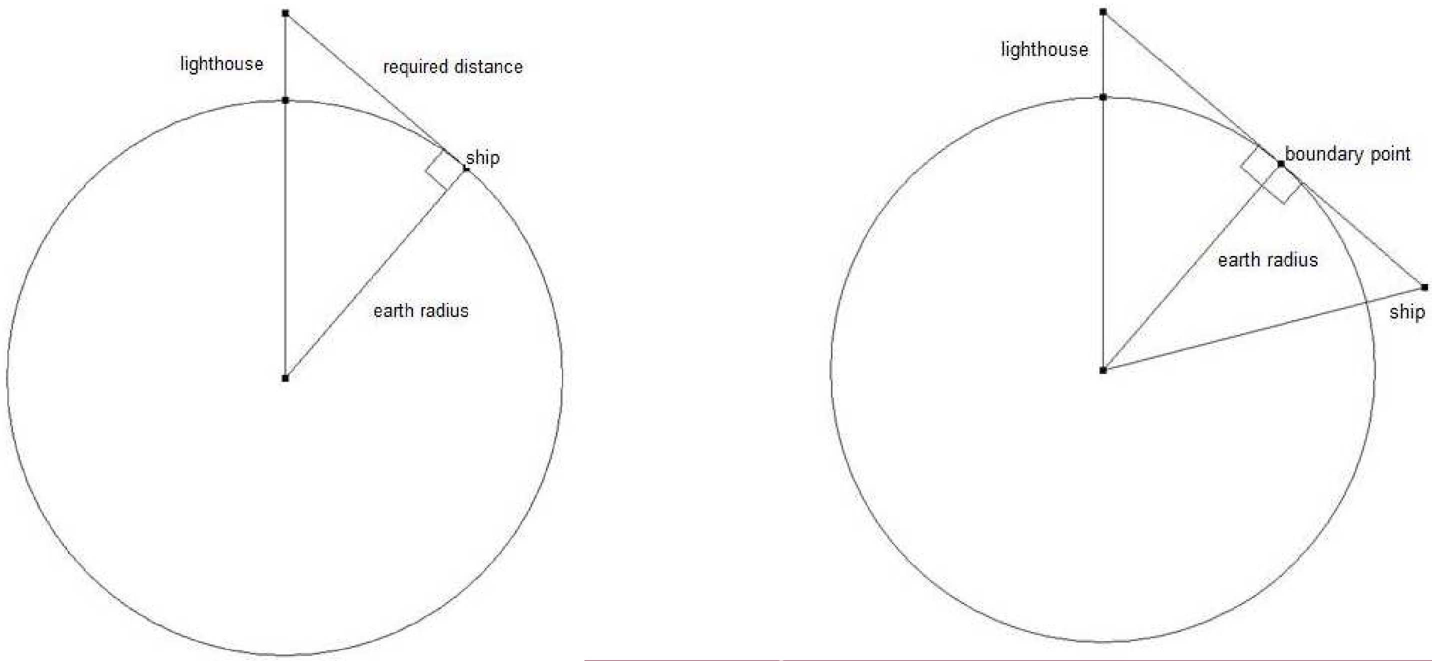
How far off the coastline was a ship when the crew was able to see the beacon of the lighthouse for the first time *(round off to whole* kilometres) ?

(original by Blum, 2006)

Real model - Simplify the situation, idealise and structure

Central idea - earth curvature and earth radius (approx. 6378.125 km) has to be taken into account.

Mathematical models



Mathematical working…

Application of Pythagorean theorem once or twice
Or: for the calculation of the real distance on earth, usage of cosine and the formula for section of a circumference

Interpreting…

Single application of the Pythagorean theorem: 19.82km, appr. 20 km

Eye height Lookout	1.7m	6.7m	11.7m	21.7m
Double usage of theorem of Pythagoras	24.45km	29.03km	32.01km	36.43km
Double usage of cosine	24.45km	29.04km	32.01km	36.43km

No important difference between distance on earth and sight line

Validating

Transformation back to reality, interpretation of the mathematical result

Goals of modelling in mathematics education

What do we want to achieve with modelling in mathematics education?

Different classifications for goals of modelling in mathematics education (e.g. Kaiser-Meßmer, 1986, Blum 1995):

- **Pedagogical goals**: imparting abilities that enable students to understand central aspects of our world in a better way;
- **Psychological goals**: fostering and enhancement of the motivation and attitude of learners towards mathematics and mathematics teaching;
- **Subject-related goals**: structuring of learning processes, introduction of new mathematical concepts and methods including their illustration;
- **Science-related goals**: imparting a realistic image of mathematics as science and into the historical relation of mathematics and its applications, supporting of critical reflections about the usage of mathematics in real world contexts.

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More recent perspectives on mathematical modelling

Distinction of different theoretical perspectives on teaching of mathematical modelling by Kaiser and Sriraman (2006):

- **Realistic or applied modelling**: goal to understand the real world and to solve real world problems important;
- **Epistemological or theoretical modelling**: promotion of the development of mathematical concepts and algorithms based on real world contexts important;
- **Educational modelling**: structuring of the learning process, fostering modelling competencies and the understanding of mathematical concepts in the foreground;
- **Model eliciting perspective**: stimulation of modelling activities through challenging real-world situations and thereby stimulate mathematical activities;
- **Socio-critical/socio-cultural modelling**: promotion of critical understanding of modelling processes and models based on recognition of cultural dependency of modelling;
- **Cognitive modelling as meta-perspective**: analysis of students' modeling processes and the cognitive and affective barriers to successful modelling as main goal.

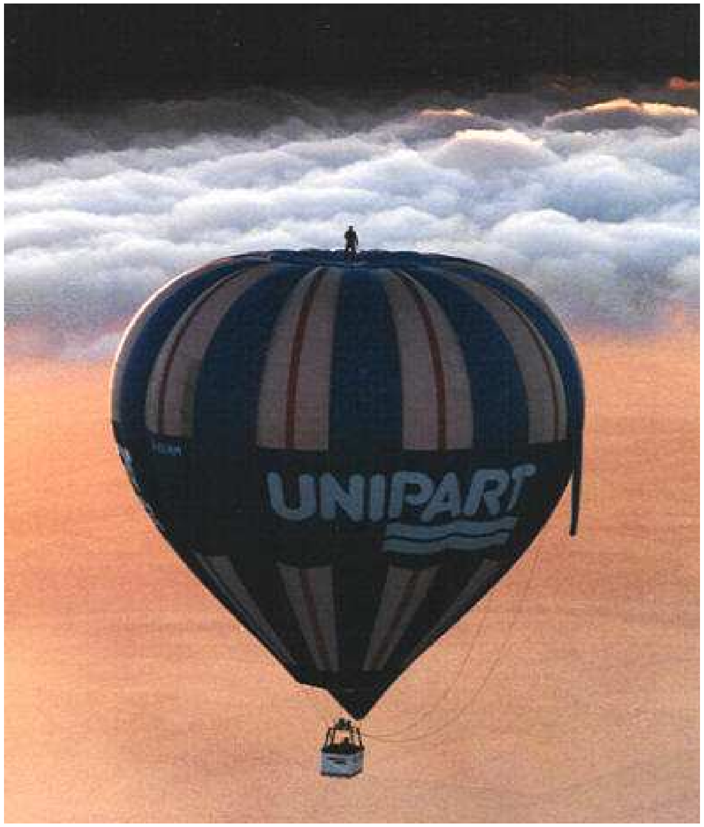
Second modelling problem

Trips with hot ail balloons are extremely popular all over the world, here is one from the balloon festival in Cumbres.

Second modelling problem

Very popular for foreign tourist seems to be hot air balloon rides in San Pedro de Atacama.

Modelling task hot air balloon



Stunt on the hot air balloon

A lot of hot air brings you up for sure. Nobody knows that better than Ian Ashpole. The 43-year-old was in England on the top of a hot air balloon. The stunt in 1500 meters height was still the safest part of the action. More critical was the start: secured only by a rope, Ashpole had to hold onto the filling balloon. During the landing, the hot air flowed out from a valve next to his legs. But except for minor burns, the balloonist received no injuries.

How many cubic meters of air the balloon holds?

Please think shortly about possible solutions, maybe together with your neighbour, two minutes.

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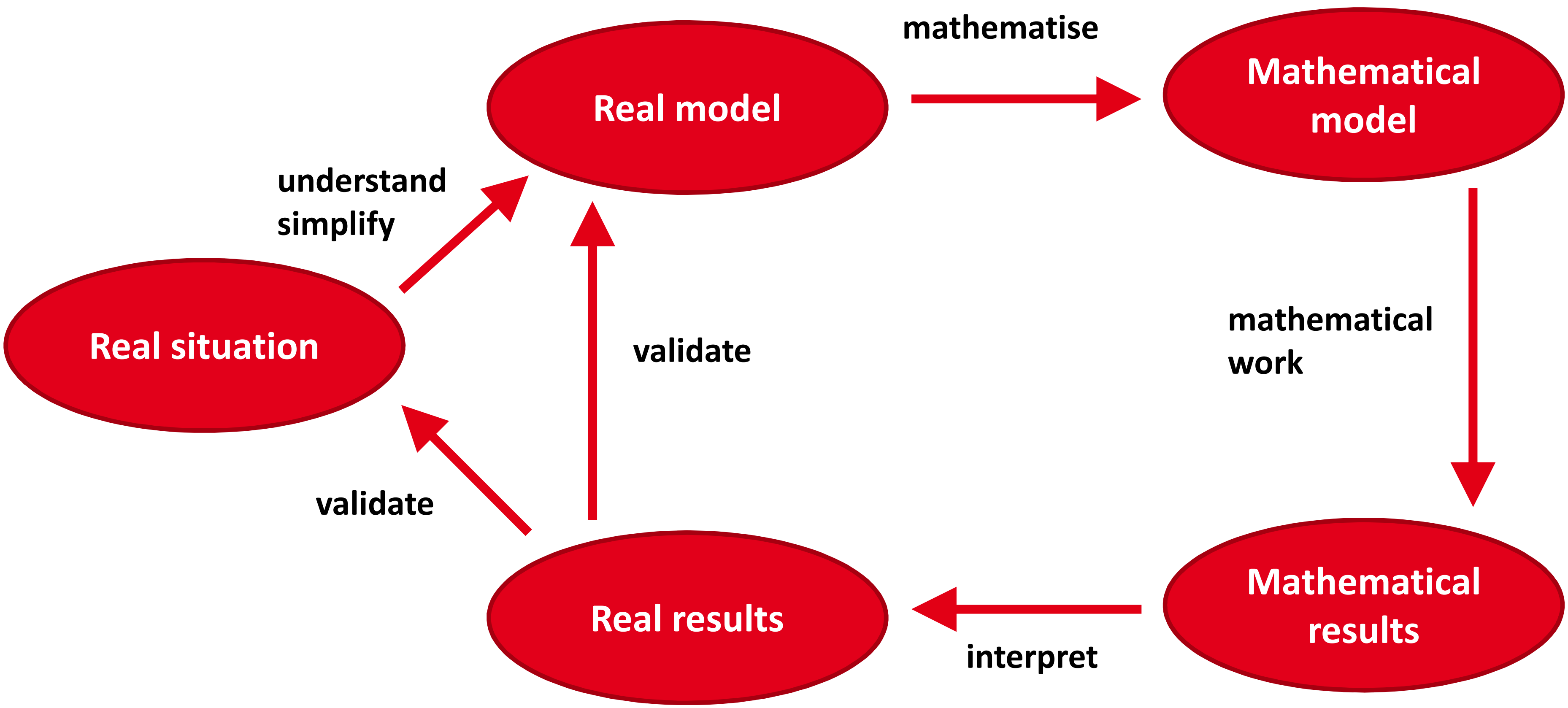
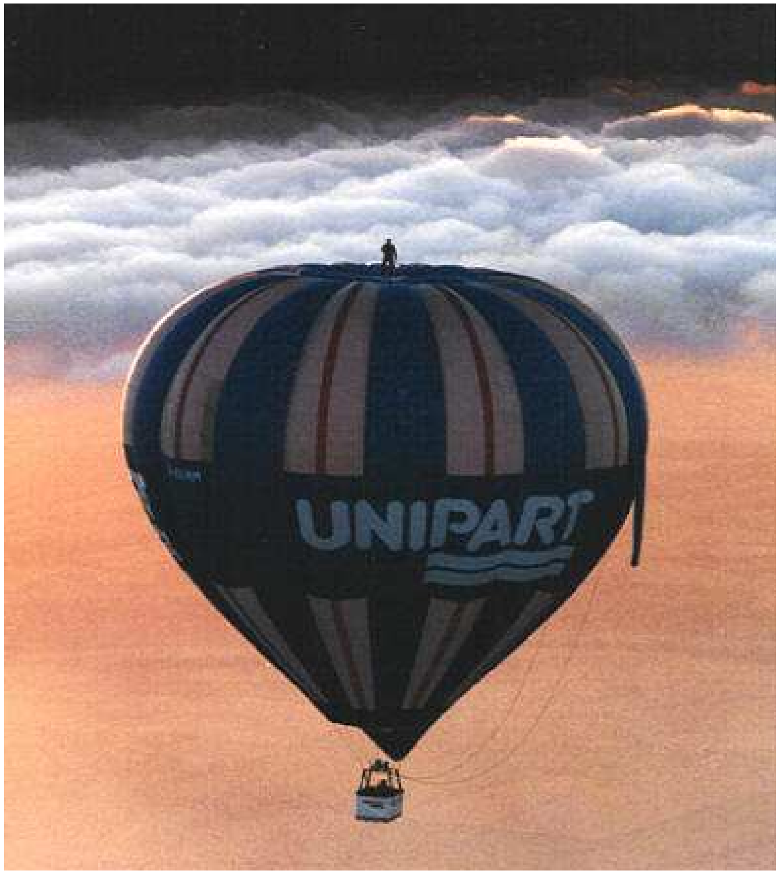
Modelling cycle (Kaiser & Stender, 2013)



Modelling problem : What is the volume of the hot air balloon?

Many ways to determine the volume of the balloon:

- First step: Using the person on the top of the balloon as scale
- Second step: Several geometrical models possible: cuboid, sphere, hemisphere, cone …; or more advanced: volume of rotation via integration



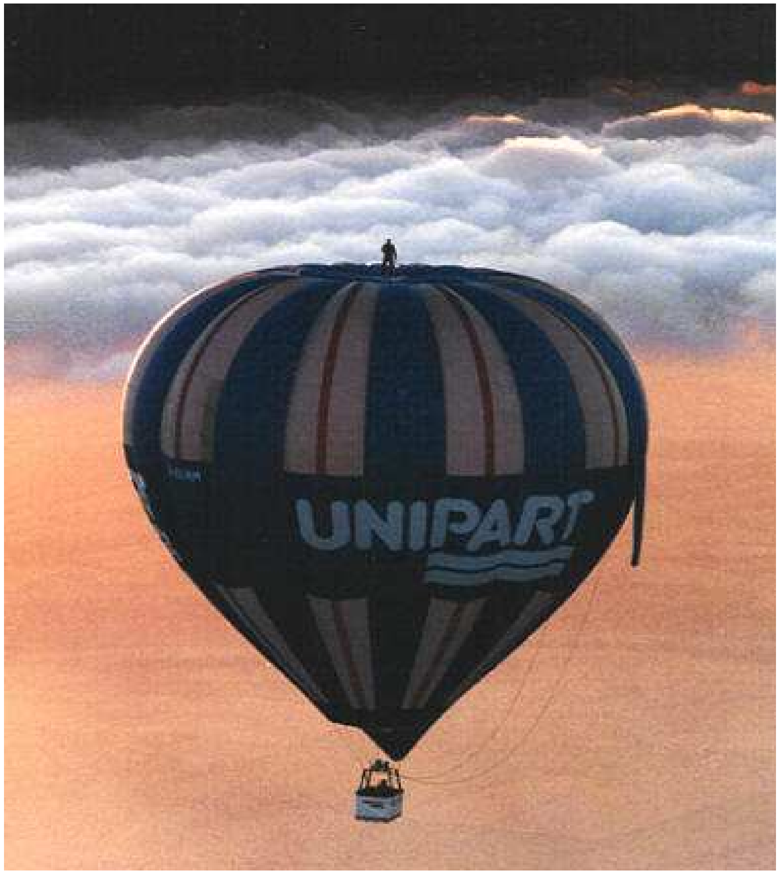
Modelling problem : What is the volume of the hot air balloon? Possible approaches

Many ways to determine the volume of the balloon:

- First step: Using the person on the top of the balloon as scale

Assumption: Ashpole 1.80 metres tall, then scale is 1:16. Height and diameter of balloon about 28.8 metres

- Second step: Selection of elementary geometrical model: sphere and combination of hemisphere and cone



Modelling problem : What is the volume of the hot air balloon?

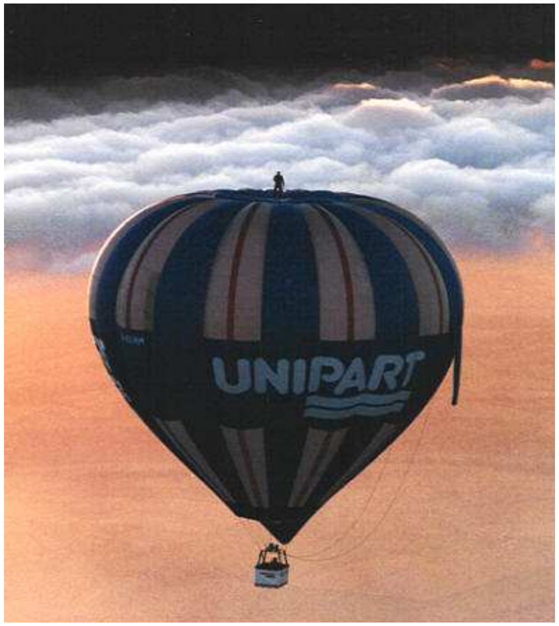
Selection of elementary geometrical model: sphere

$$V_{sphere} = \frac{4}{3} \cdot \pi \cdot r^3 = \frac{4}{3} \cdot \pi \cdot 14.4^3 = \frac{4}{3} \cdot \pi \cdot 2985.984 = 3981.12 \cdot \pi$$

$$V_{sphere} = 3981.312 \cdot \pi \approx 12500 \text{ m}^3$$

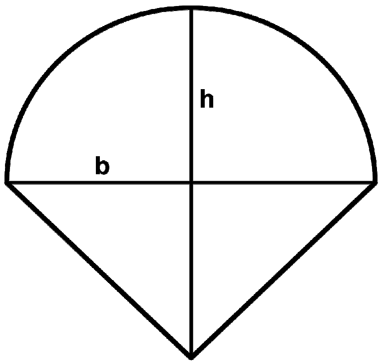
Validation via Internet search:

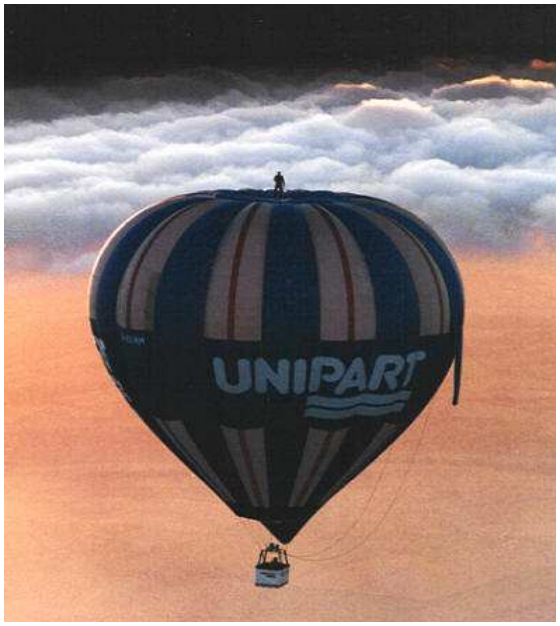
FAI-Klasse	Nenninhalt
AX 3	401 - 600 m^3
AX 4	601 - 900 m^3
AX 5	901 - 1200 m^3
AX 6	1201 - 1600 m^3
AX 7	1601 - 2200 m^3
AX 8	2201 - 3000 m^3
AX 9	3001 - 4000 m^3
AX 10	4001 - 6000 m^3
AX 11	6001 - 9000 m^3
AX 12	9001 - 12000 m^3



Modelling problem : What is the volume of the hot air balloon?

Refinement: semi-sphere and cone



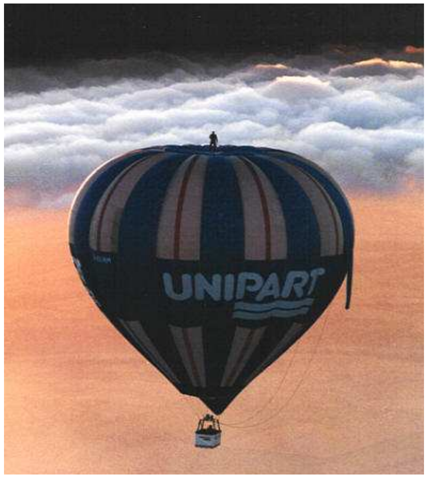


$$V_{semisphere} = \frac{2}{3} \cdot \pi \cdot r^3 \approx 6{,}250$$

$$V_{cone} \frac{1}{3} \cdot \pi \cdot r^2 \cdot \frac{h_{total}}{2} \approx 1{,}550$$

$$V_{balloon} = V_{semisphere} + V_{cone} \approx 6250 + 1550 \approx 7800 \text{ m}^3$$

Modelling problem : What is the volume of the hot air balloon?



$$V_{balloon} = V_{semisphere} + V_{cone} \approx 6250 + 1550 \approx 7800 \text{ m}^3$$

Interpretation phase:

New result more in line with the results found in the internet.

Implementation in school

Problem not easy for students due to ist under-determined nature, many parameters missing, which must be conjectured.

Video from one group of three boys, students from higher track school, so-called Gymnasium, in Hamburg, strong group in their metacognitive behaviour, but still had many difficulties at the beginning (Vorhölter, Krüger, & Wendt, 2019).

Third modelling Example – Uwe Seeler's foot

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Uwe Seeler's foot

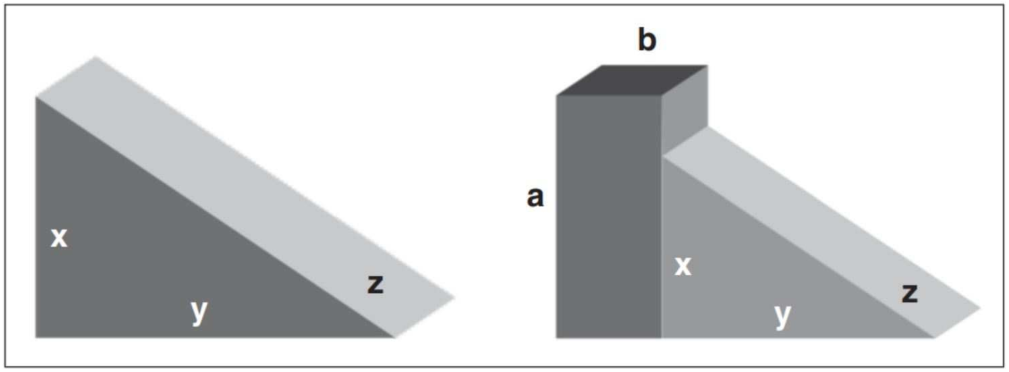
Since August 2005, there has been a sculpture of the right foot of Uwe Seeler, a famous German soccer player, in front of the football arena in Hamburg, Germany. A newspaper, the *Hamburger Abendblatt*, reported that Uwe Seeler's real foot fits exactly 3,980 times into the sculpture.

Is it possible? Uwe Seeler's shoe size is 10½.

EXAMPLE – UWE SEELER'S FOOT

- Simplify/ Understand:
  - What is the goal?
    - two ideas: hollow sculpture that has to be filled with shoes or comparison of volumes
  - Assumptions:
    - The women‘s height is estimated.
- Mathematise:
  - Using this to find out the scale, it can be assumed that the sculpture is 5.15m long, 3.50m high und 2.30m wide.
  - Research: A feet with a shoe size of 10.5 is about 27cm long,17.5 cm high and 11.5cm wide.

EXAMPLE – UWE SEELER'S FOOT



- Deciding for a geometric object (e.g. figure on the left)
- Mathematical work:
  - Calculating both volumes
  - Comparing both volumes
- Interpretation: The foot fits ca. 7732 times into the sculpture. Therefore the statement in the newspaper cannot be true.
- Validation of the model: Is there a better model? Would it be more suitable to split it into several geometric objects?

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How do we model – Modelling cycles

Approaches to mathematical modelling – modelling cycles

How do we model?

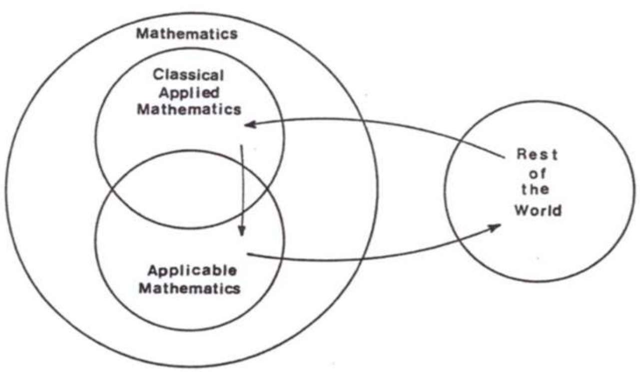
Several approaches to describe modelling processes

Well accepted are **modelling cycles** with different **stages** departing from the real world problem and coming back to its solution via mathematical models (Kaiser, Blum, Maaß, Stillman, Galbraith ….)

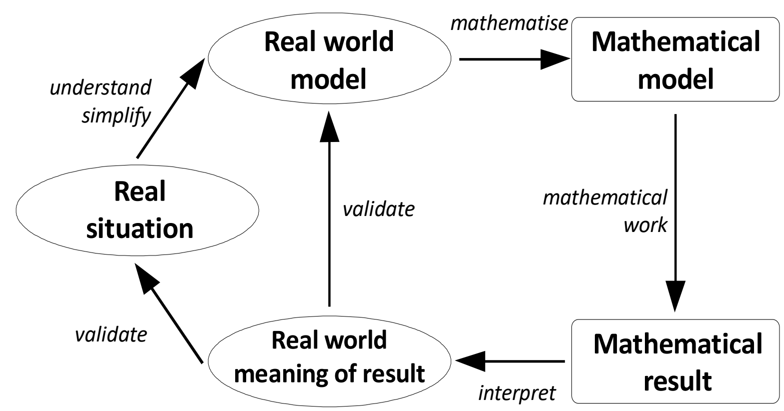
Modelling cycles

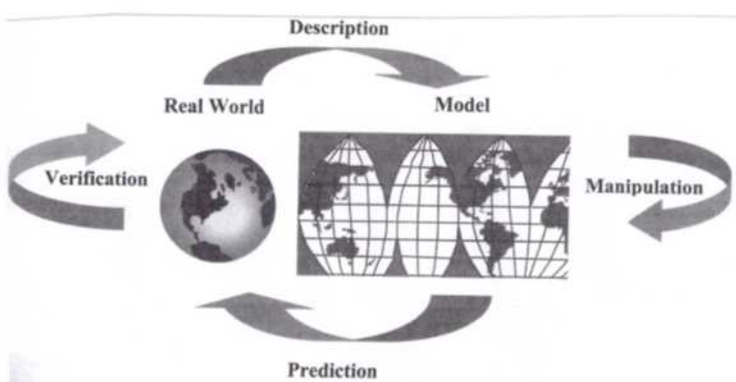
Different kinds of modelling cycles developed by these perspectives.

Modelling cycle from applied mathematics (Pollak, 1979)



Modelling cycle from educational modelling (Kaiser & Stender, 2013)





Cycle from Model eliciting perspective (Lesh & Doerr, 2003)

2. Modelling competencies: definition, measurement and fostering

Volume 109, issue 2, February 2022

Innovations in measuring and fostering modelling competencies

Issue editors
Gabriele Kaiser & Stanislaw Schukajlow

11 articles in this issue

Cevikbas, M., Kaiser, G., & Schukajlow, S. (2022). **A systematic literature review of the current discussion on mathematical modelling competencies: state-of-the-art developments in conceptualizing, measuring, and fostering**. *Educational Studies in Mathematics,* 109, 205-236.

Definition of mathematical modelling competencies – Systematic literature review

Distinction of top-down versus bottom-up approach (Niss & Blum, 2020):

- **Top-down or holistic approach**: definition of modelling competency as a holistic construct, as one of eight mathematical competencies, which is not further differentiated; definition as someone‘s insightful readiness to act in response to the challenges of a given situation (Niss, 1999);
- **Bottom-up or analytic approach**: definition of modelling competence as the ability of a person to solve a real world problem with mathematical methods and the willingness to do so (volition); clear separation from abilities (Kaiser, 2007; Maass, 2007); distinction of:
  - **sub-competencies** defined along the modelling cycle, e.g. competencies to understand real world problem and construct a real-world model;
  - **overall competence** to carry out a modelling process
  - **additional competencies** such as **metacognitive** competencies, **communication** competencies …

Currently, **dominance of the bottom-up approach in empirical research** according to systematic literature survey. Overall, no newer theoretical foundation, sign of some kind of saturation of the discourse.

Measurement of modelling competencies - Systematic literature review

Great variety of measurement instruments for modelling competencies.

- **Dominance of informal test instruments** (not standardised), followed by audio-video and screen recordings;
- Frequent usage of **written reports** such as project reports, retrospective reports, written task formulations; furthermore usage of questionnaires and interviews;
- Quite **seldom usage of observations** or oral examinations/presentations, or worksheets.

Partly reference to established instruments, seldom development of new standardised instruments. Clear need of standardised instruments as developed by Haines, Izard, Houston twenty years ago.

As results based on systematic literature review focused on papers in proceedings (ICTMA conference series) and in indexed journals written in English, some kind of **distortion**, e.g. slight dominance of quantitatively oriented studies against qualitatively oriented studies.

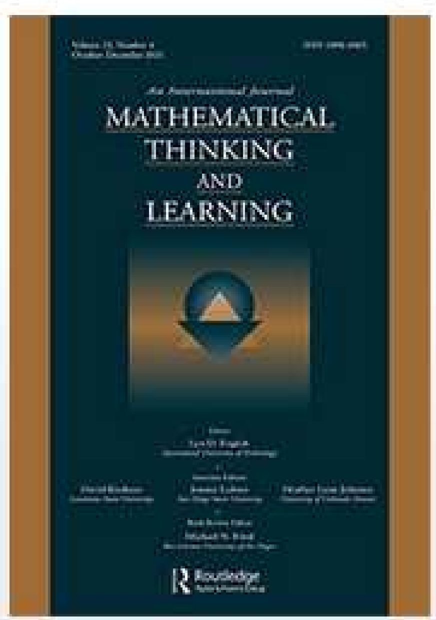
Fostering of modelling competencies - Systematic literature review

Variety of activities to foster modelling competencies.

- **Many activities of designing and conducting training strategies** (modelling course/seminar, professional development program, projects, teaching units);
- Furthermore, **activities concerning exposing to modelling tasks** (sequences) activities and gaining experience in modelling;
- Finally, activities focusing psychological factors such as promoting motivation, metacognitive factors such as promoting metacognitive awareness, use of digital tools/technologies.

Fostering activities not mentioned in many papers. However, same limitation due to the study method.

3. Further results of empirical studies: cognitive, meta-cognitive and affective aspects; teacher interventions and adaptive teacher support; new media



Special issue of *Mathematical Thinking and Learning* on „Modelling from a cognitive perspective“ edited by Kaiser, Schukajlow, & Stillman (2023)

Survey paper on „Modelling from a cognitive perspective: theoretical considerations and empirical contributions“ by guest editors

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Cognitive, metacognitive and affective aspects

Wealth of empirical studies, qualitatively or quantitatively oriented:

Empirically shown that each step within the modelling process represents a potential cognitive barrier for students (for an overview see Blum 2011, 2015):

- Prominently, Stillman et al. (2010) and Galbraith et al. (2007) with a theoretical approach for the **analysis of learners' metacognitive processes** on these potential "**blockages**" or "**red flag situations**": no progress made or errors not recognized;
- **Necessity of using metacognitive activities**, especially **reflective activities** during transition phases such as analysing the course/phase of the modelling process, checking the individual steps and referring back to the original problem; in addition **modelling cycles as metacognitive measure** for supporting modelling activities (Matos & Carreira, 1997; Stillman, 2011; Schukajlow et al. 2012; Stender, 2019; Vorhölter et al. 2019).

Cognitive, metacognitive and affective aspects

Wealth of empirical studies, qualitatively or quantitatively oriented:

- Occurrence of **individual modelling routes** by the learners with mini-cycles jumping back and forth between modelling phases, not following the ideal-typical modelling cycles (Borromeo Ferri 2011);
- **Personal significance** that learners give to working on modelling tasks as influencing factor (Vorhölter, 2009);
- Different **preferences for modelling** within learners: not interested in modelling / not interested in real world / not interested in mathematics / not interested in mathematics and real world (Maaß, 2004);
- Different **preferences for dealing with real world contexts**: ambivalent relationship as well as a positively integrating or rejecting approach (Busse, 2009).

Students' difficulties in the modelling process – A Video-vignette

Context: The students, attending class 9 of a German higher track school (age 14-15), are working on the problem of Uwe Seeler's foot.

Please try while watching the video

- **a) to identify, which difficulties the students' encounter,**
- **b) which metacognitive means could help students to overcome these problems**

This is a scripted video but based on real videotaped modelling processes (Vorhölter, Krüger, & Wendt, 2019).


Modellierungskreislauf
Uwe Seeler's foot. Since August 2005 a sculpture of Uwe Seeler's right foot,
a famous German soccer player, can be seen in front of the football arena
in Hamburg.

students' difficulties

Students' difficulties in the video-vignette:

1. Solving under-determined tasks (metacognitive knowledge): making assumptions: "*What does that mean? What are we supposed to do?"; "But we can't solve the questions without numbers, so we're done"*
2. Unsuitable use of routines to solve a problem without considering the context: "W*e have 3980 and 42. We could do something with that, right?*"; *"We could just divide 3980 by 42. Then we know how often his foot fits into the sculpture."*
3. Extra-mathematical knowledge: "*Uwe, has a shoe size of 42. Um, I don't really know, what is that in cm?" – "42cm"*

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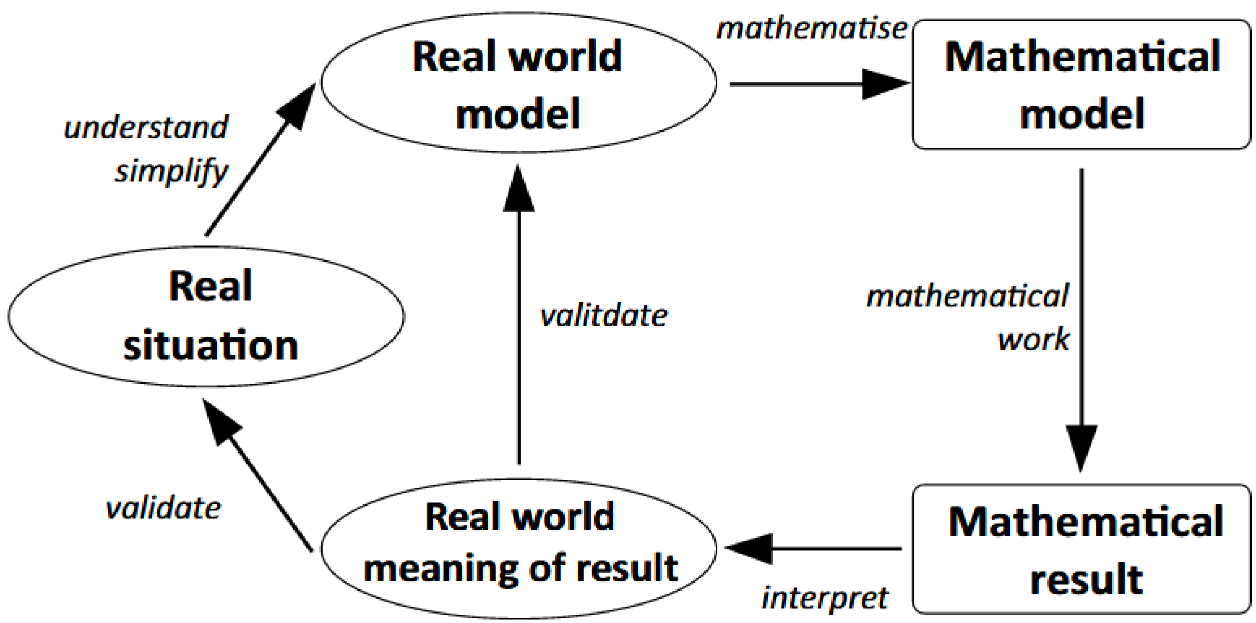
STUDENTS‘ DIFFICULTIES

- Dominant phase of the modelling cycle creating difficulties: **Understand**/ **simplify**: “*What are we supposed to do?“;* “I’d rather say we have to confirm that the foot fits into it 3980 times“



Kaiser & Stender 2013

STUDENTS‘ DIFFICULTIES

- Metacognition:
  - **Planning**: “*Well, let’s just think about what we need and what we already know*?“; Planning different ways to solve the problem (end of the video)
    → tackle difficulty (1)
  - **Monitoring**: “*But what does that mean for the question?*?“
    → tackle difficulty (2)
  - **Regulation**: Measuring their own shoe to get an idea of shoe sizes
    → tackle difficulty (3)

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Teacher interventions and adaptive teacher support

Variety of empirical studies, qualitatively or quantitatively oriented:

- Consensus about necessity for own activities: **Modelling is not a spectator sport** (Helmut Neunzert, ICTMA14, 2009);
- Central influence of **adaptive teacher support** and potential of **independent individual work within in a group** on cognitive and motivational aspects of modelling tasks: higher modelling performances of learning in "operative-strategic" design (more independence-oriented) compared to the "directive" design (more teacher-centred) and better development of self-regulation (Schukajlow et al., 2012, Blum & Schukajlow, 2018; Durandt, Blum, & Lindl, 2022);
- Positive influence of fostering **multiple solution methods** on modelling achievements as well as drawing diagrams and making sketches (Krug & Schukajlow, 2020; Achmetli et al., 2019);

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Teacher interventions and adaptive teacher support

Variety of empirical studies, qualitatively or quantitatively oriented:

- Necessity of **careful diagnosis** within the modelling activities focusing students' potential and difficulties (Blum, 2011, 2015), reference to general approach of **scaffolding**.
- Usage of a **taxonomy of interventions** in mathematical modelling - becoming more content-specific and more directive within the process; hardly any **strategic interventions** by teachers within modelling activities (Leiß, 2007; Stender, 2016); development **of metacognitive activities** of students by fostering of adaptive heuristic strategies and strategic interventions (Kaiser & Stender, 2013);

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Table of possible teacher interventions

	Motivational help	Feedback help	General-strategic help	Content-oriented strategic help	Content help
In all modelling phases	You can solve the problem	You are on the right way.	Which part of the modelling cycle are you currently working on?		
Formulate real world model			Have you simplified the task sufficiently?		
Translate real world model in mathematical model			Which mathematical methods might be appropriate?		
Solve mathematical model				How about trying to find a formula (if possible) ?	Have you checked this step?
Interpret the mathematical results, validate the results in real world situations			Think about which assumptions you could change to adjust your model to reality.	What happens in extreme cases?	

New media

Currently not many empirical studies:

- Differences in the **promotion of certain sub-competences** of modelling by intervention studies with dynamic geometry software or with conventional means: positive relation of **program-related self-efficacy** expectations and attitudes towards the software and program-related self-efficacy expectancy as significant predictor of mathematisation performance (Greefrath et al., 2018; Hankeln & Greefrath, 2020).
- Usefulness of **digital tools for each step of the modelling process** (Geiger, 2011; Greefrath & Siller, 2018).
- Necessity of **significant support of the learners** to use digital tools in a variety of ways when modelling - especially beyond mathematical work (Geiger et al., 2003). Unfortunately, no usage of digital tools within work on reality-based tasks, such as the graphical possibilities to represent or compare the chosen mathematical models (Brown, 2015).

4. Looking ahead

Instead of a summary - Prologue by David Burghes Proceedings ICTMA1, 1983, Exeter

„The basic philosophy behind the approach … of the modelling workshop for higher education is that to become proficient in modelling, you must fully experience it – it is no good just watching somebody else do it, or repeat what somebody else has done – you must experience it yourself. I would liken it to the activity of swimming. You can watch others swim, you can practice exercises, but to swim, you must be in the water doing it yourself.“

Looking ahead or What is Needed in the next years:

- **Need of theoretical work:** further development of central constructs such as modelling competencies;
- Development of **standardised measurement instruments** and encouragement of **exchange of measurement instruments** within the modelling community (similar to the tests by Haines et al.);
- Inclusion of studies with **mixed-methods design:** qualitatively oriented in-depth study accompanying quantitative studies and more quantitatively oriented "larger-scale" studies;
- **Scaling-up of established learning environments** within controlled implementations, e.g. by laboratory studies.

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References

Achmetli, K., Schukajlow, S. & Rakoczy, K. (2019). Multiple Solutions for Real-World Problems, Experience of Competence and Students' Procedural and Conceptual Knowledge. International Journal of Science and Mathematics Education, 17, 1605–1625

Alwast, A., & Vorhölter, K. (2022). Measuring pre-service teachers' noticing competencies within a mathematical modeling context – analysis of an instrument. Educational Studies in Mathematics, 109, 237-250.

Barzel, B., Hußmann, S., & Leuders, T. (2005). Computer, Internet & Co im Mathematikunterricht. Berlin: Cornelsen Scriptor.

Bimba, A. T., Idris, N., Al-Hunaiyyan, A., Mahmud, R. B., & Shuib, N. L. B. M. (2017). Adaptive feedback in computer-based learning environments: A review. Adaptive Behavior, 25(5), 217–234. https://doi.org/10.1177/1059712317727590

Blomhøj, M., & Jensen, T. H. (2003). Developing mathematical modelling competence: Conceptual clarification and educational planning. Teaching Mathematics and its Applications, 22(3), 123–139.

Blum, W. (1985). Anwendungsorientierter Mathematikunterricht in der didaktischen Diskussion. Mathematische Semesterberichte, 32(2), 195–232.

Blum, W. (1996). Anwendungsbezüge im Mathematikunterricht – Trends und Perspektiven. In G. Kadunz, H. Kautschitsch, G. Ossimitz, & E. Schneider (ed.), Trends und Perspektiven (pp. 15–38). Wien: Hölder-Pichler-Tempsky.

Blum, W. (2011). Can modelling be taught and learnt? Some answers from empirical research. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (ed.), Trends in teaching and learning of mathematical modelling (pp. 15–30). Dordrecht: Springer.

Blum, W. (2015). Quality teaching of mathematical modelling: What do we know, what can we do? In S. J. Cho (Ed.), The Proceedings of the 12th International Congress on Mathematical Education (pp. 73-96). Cham: Springer.

Blum, W. (2006). Modellierungsaufgaben im Mathematikunterricht – Herausforderung für Schüler und Lehrer In A. Büchter et al. (eds.), Realitätsnaher Mathematikunterricht – vom Fach aus und für die Praxis (pp. 8–23). Hildesheim: Franzbecker.

Blum, W., & Leiß, D. (2005). Modellieren im Unterricht mit der „Tanken“-Aufgabe. mathematik lehren, 128, 18–21.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects – state, trends and issues in mathematics instruction. Educational Studies in Mathematics, 22, 37–68.

Blum, W., Galbraith, P.L., Henn, H.-W., & Niss, M. (ed.) (2007). Modelling and applications in mathematics education. The 14th ICMI study. New York: Springer.

Blum, W., & Schukajlow, S. (2018). Selbständiges Lernen mit Modellierungsaufgaben – Untersuchung von Lernumgebungen zum Modellieren im Projekt DISUM. In S. Schukajlow & W. Blum (ed.), Evaluierte Lernumgebungen zum Modellieren (pp. 51–72). Springer Fachmedien Wiesbaden.

Borromeo Ferri, R. (2011). Wege zur Innenwelt des mathematischen Modellierens. Wiesbaden: Vieweg + Teubner.

Borromeo Ferri, R. (2018). Learning how to teach mathematical modelling in school and teacher education. Cham: Springer.

Borromeo Ferri, R. (2021). Mandatory mathematical modelling in school: What do we want the teachers to know? In F. K. S. Leung, G. A. Stillman, G. Kaiser, K. L. Lok (ed.), Mathematical Modelling Education in East and West (S. 103-117). Cham: Springer.

Borromeo Ferri, R., & Blum, W. (2010). Mathematical Modelling in Teacher Education – Experiences from a Modelling Seminar. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (ed.), Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education (S. 2046–2055). Lyon: Institut National De Recherche Pédagogique.

Borromeo Ferri, R., & Blum, W. (ed.). (2018). Lehrerkompetenzen zum Unterrichten mathematischer Modellierung: Konzepte und Transfer. Springer Fachmedien Wiesbaden. https://doi.org/10.1007/978-3-658-22616-9

Borromeo Ferri, R., Kaiser, G., & Blum, W. (2011). Mit dem Taxi durch die Welt des mathematischen Modellierens. In T. Krohn, E. Malitte, G. Richter, K. Richter, S. Schöneburg, & R. Sommer (ed.), Mathematik für alle. Wege zum Öffnen von Mathematik – Mathematikdidaktische Ansätze. Festschrift für Wilfried Herget (S. 35–47). Hildesheim: Franzbecker.

Borromeo Ferri, R., Kaiser, G., & Paquet, M. (2021, accepted). Meeting the challenge of heterogeneity through the self-differentiation potential of mathematical modeling problems. In R. Leikin (ed.), Mathematical challenge for all. Cham: Springer.

Bracke, M., Ludwig, M., & Vorhölter, K. (2021, im Druck). Neue Materialien für einen realitätsbezogenen Mathematikunterricht, Band 8. Wiesbaden: Springer Fachmedien. https://doi.org/10.1007/978-3-658-33012-5_5

Brand, S. (2014). Erwerb von Modellierungskompetenzen. Ein empirischer Vergleich eines holistischen versus eines atomistischen Modellierungsansatzes zur Förderung von Modellierungskompetenzen. Wiesbaden: Springer Spektrum.

Bräuer, V., Leiß, D., & Schukajlow, S. (2021). Skizzen zeichnen zu Modellierungsaufgaben – Eine Analyse themenspezifischer Differenzen einer Visualisierungsstrategie beim mathematischen Modellieren. Journal für Mathematik-Didaktik, https://doi.org/10.1007/s13138-021-00182-7.

Brown, J. P. (2015). Visualisation Tactics for Solving Real World Tasks. In G. Stillman, W. Blum, & M. S. Biembengut (ed.), Mathematical Modelling in Education Research and Practice (S. 431–442). Springer.

Buchholtz, N. (2018). Wie können Lehrkräfte Mathematisierungskompetenzen bei Schülerinnen und Schülern fördern und diagnostizieren? In R. Borromeo Ferri, & W. Blum (ed.), Lehrerkompetenzen zum Unterrichten mathematischer Modellierung, (S. 57-80). Cham: Springer.

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Burkhardt, H. (1981). The Real World and Mathematics. Glasgow: Blackie.

Burkhardt, H., & Pollak, H.O. (2006). Modelling in mathematics classrooms: Reflections on past developments and the future. ZDM – Mathematics Education,38(2), 178–195.

Busse, A. (2009). Umgang Jugendlicher mit dem Sachkontext realitätsbezogener Mathematikaufgaben. Ergebnisse einer empirischen Studie. Hildesheim: Franzbecker.

Cevikbas, M., Kaiser, G., & Schukajlow, S. (2022). A systematic literature review of the current discussion on mathematical modelling competencies: State-of-the-art developments in conceptualizing, measuring, and fostering. Educational Studies in Mathematics, 109, 205-236 .

Drijvers, P., & Doorman, M. (1996). The graphics calculator in mathematics education. The Journal of Mathematical Behavior, 15(4), 425–440.

Durandt, R., Blum, W., & Lindl, A. (2021). Fostering Mathematical Modelling Competency of South African Engineering Students: Which Influence does the Teaching Design have? Educational Studies in Mathematics, https://doi.org/10.1007/s10649-021-10068-7

Frenken, L., Greefrath, G., Siller, H.-S., & Wörler, J. F. (2021). Analyseinstrumente zum mathematischen Modellieren mit digitalen Medien und Werkzeugen. mathematica didactica, 44(1).

Freudenthal, H. (1968). Why to teach mathematics so as to be useful. Educational Studies in Mathematics, 1(1/2), 3–8.

Freudenthal, H. (1973). Mathematics as an educational task. Dordrecht: Reidel.

Galbraith, P., Stillman, G., Brown, J. & Edwards (2007). Facilitating middle secondary modeling competencies. In C. P. Haines, P. Galbraith, W. Blum, & S. Khan (ed.), Mathematical modelling (ICTMA 12): Education, engineering and economics (S. 130–141). Chichester: Horwood.

García, F. J., Maass, K., & Wake, G. (2010). Theory meets practice: Working pragmatically within different cultures and traditions. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (ed.), Modeling students' mathematical modeling competencies. ICTMA13 (S. 445–457). New York: Springer.

Geiger, V. (2011). Factors affecting teachers' adoption of innovative practices with technology and mathematical modelling. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (ed.), Trends in teaching and learning of mathematical modelling (S. 305–314). Springer Netherlands. https://doi.org/10.1007/978-94-007-0910-2_31

Geiger, V., Galbraith, P., Renshaw, P., & Goos, M. (2003). Choosing and Using Technology for Secondary Math-ematical Modelling Tasks – Choosing the Right Peg for the Right Hole. In Mathematical Modelling in Education and Culture (S. 126–140). Elsevier. https://doi.org/10.1533/9780857099556.3.126

Goos, M. (2002). Understanding metacognitive failure. Journal of Mathematical Behavior, 21(3), 283–302.

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Greefrath, G., Kaiser, G., Blum, W., & Borromeo Ferri, R. (2013). Mathematisches Modellieren – Eine Einführung in theoretische und didaktische Hintergründe. In R. Borromeo Ferri, G. Greefrath, & G. Kaiser (ed.), Mathematisches Modellieren für Schule und Hochschule (pp. 11–37). Springer Fachmedien Wiesbaden.

Greefrath, G., & Siller, H.-S. (2018a). Digitale Werkzeuge, Simulationen und mathematisches Modellieren. In G. Greefrath & H.-S. Siller (ed.), Digitale Werkzeuge, Simulationen und mathematisches Modellieren (S. 3–22). Springer Fachmedien Wiesbaden. https://doi.org/10.1007/978-3-658-21940-6_1

Greefrath, G., & Siller, H.-S. (2018b). GeoGebra as a Tool in Modelling Processes. In L. Ball, P. Drijvers, S. Ladel, H.-S. Siller, M. Tabach, & C. Vale (ed.), Uses of Technology in Primary and Secondary Mathematics Education (S. 363–374). Springer International Publishing.

Greefrath, G., Siller, H.-S., Klock, H., & Wess, R. (2021). Pre-service secondary teachers' pedagogical content knowledge for the teaching of mathematical modelling. Educational Studies in Mathematics, https://doi.org/10.1007/s10649-021-10038-z

Greefrath, G., & Weigand, H.-G. (2012). Simulieren – mit Modellen experimentieren. mathematik lehren, 174, 2–6.

Greefrath, G., Hertleif, C., & Siller, H.-S. (2018). Mathematical modelling with digital tools—A quantitative study on mathematising with dynamic geometry software. ZDM Mathematics Education, 50(1–2), 233–244:

Gurjanow, I., Jablonski, S., Ludwig, M., & Zender, J. (2019). Modellieren mit MathCityMap: Praxisbezogene Beispiele zum Modellieren am realen Objekt. In I. Grafenhofer & J. Maaß (ed.), Neue Materialien für einen realitätsbezogenen Mathematikunterricht 6 (S. 95–105). Springer Fachmedien Wiesbaden:

Haines, C. R., Crouch, R. M., & Davis, J. (2000). Understanding students' modelling skills. In J. F. Matos, W. Blum, K. Houston, & S. Carreira (ed.), Modelling and mathematics education: ICTMA9 applications in science and technology (S. 366–381). Chichester: Horwood.

Hankeln, C., & Greefrath, G. (2020). Mathematische Modellierungskompetenz fördern durch Lösungsplan oder Dynamische Geometrie-Software? Empirische Ergebnisse aus dem LIMo-Projekt. Journal für Mathematik-Didaktik. https://doi.org/10.1007/s13138-020-00178-9

Henn, H.-W. (1998). The impact of computer algebra systems on modelling activities. In P. Galbraith, W. Blum, G. Book, & I.D. Huntley (ed.), Mathematical Modelling. Teaching and Assessment in a technology-rich world (S. 115–124). Horwood.

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Hischer, H. (2016). Mathematik – Medien – Bildung. Springer Fachmedien Wiesbaden. https://doi.org/10.1007/978-3-658-14167-7

Hoyles, C., & Lagrange, J.-B. (ed.). (2010). Mathematics education and technology: Rethinking the terrain: the 17th ICMI study. Springer.

Kaiser-Meßmer, G. (1986). Anwendungen im Mathematikunterricht. Bd. 1 – Theoretische Konzeptionen. Bd. 2 – Empirische Untersuchungen. Bad Salzdetfurth: Franzbecker.

Kaiser, G. (1995). Realitätsbezüge im Mathematikunterricht – Ein Überblick über die aktuelle und historische Diskussion. In G. Graumann, T. Jahnke., G. Kaiser, & J. Meyer (ed.), Materialien für einen realitätsbezogenen Mathematikunterricht (S. 66–84). Bad Salzdetfurth: Franzbecker.

Kaiser, G. (2007). Modelling and Modelling Competencies in School. In C. Haines, P. L. Galbraith, W. Blum, & S. Khan (ed.), Mathematical Modelling (ICTMA 12): Education, Engineering and Economics (S. 110–119). Horwood.

Kaiser, G. (2017). The Teaching and Learning of Mathematical Modeling. In J. Cai (ed), Compendium for Research in Mathematics Education (S. 267-291). Reston, VA: National Council of Teachers of Mathematics.

Kaiser, G. (2007). Modelling and modelling competencies in school. In C. Haines, P. Galbraith, W. Blum, & S. Khan (ed.), Mathematical modelling (ICTMA 12): Education, engineering and economics (S. 110–119). Chichester: Horwood.

Kaiser, G., & Schwarz, B. (2010). Authentic modelling problems in mathematics education – Examples and experiences. Journal für Mathematik-Didaktik, 31(1), 51–76.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modeling in mathematics education. ZDM Mathematics Education, 38(3), 302–310.

Kaiser, G., & Stender, P. (2013). Complex modelling problems in co-operative, self-directed learning environments. In G. Stillman, G. Kaiser, W. Blum, & J. Brown (ed.), Teaching mathematical modelling: Connecting to research and practice (pp.. 277–293). Dordrecht: Springer.

Kaiser, G., Bracke, M., Göttlich, S., & Kaland, C. (2013). Realistic complex modelling problems in mathematics education. In R. Strässer & A. Damlamian (ed.), Educational interfaces between mathematics and industry (20th ICMI study) (pp. 287–297). New York: Springer.

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Krawitz, J., Chang, Y.-P., Yang, K.-L., Schukajlow, S. (2021). The role of reading comprehension in mathematical modelling: improving the construction of a real-world model and interest in Germany and Taiwan. Educational Studies in Mathematics, https://doi.org/10.1007/s10649-021-10058-9.

Krug, A., & Schukajlow, S. (2020). Entwicklung prozeduraler Metakognition und des selbstregulierten Lernens durch den Einsatz multipler Lösungen zu Modellierungsaufgaben. Journal für Mathematik-Didaktik, 41, 423–458, https://doi.org/10.1007/s13138-019-00154-y

Leiß, D. (2007). Lehrerinterventionen im selbständigkeitsorientierten Prozess der Lösung einer mathematischen Modellierungsaufgabe. Hildesheim: Franzbecker.

Leiß, D., Schukajlow, S., Blum, W., Messner, R., & Pekrun, R. (2010). The role of the situation model in mathematical modelling – task analyses, student competencies, and teacher interventions. Journal für Mathematikdidaktik, 31(1), 119-141

Lesh, R., & Doerr, H. (ed.) (2003). Beyond constructivism: models and modeling perspectives on mathematics problem solving, learning, and teaching. Mahwah: Lawrence Erlbaum.

Maaß, K. (2004). Mathematisches Modellieren im Unterricht: Ergebnisse einer empirischen Studie. Hildesheim: Franzbecker.

Maaß, K. (2006). What are modelling competencies? Zentralblatt für Didaktik der Mathematik, 38(2), 113–142.

Maaß, K., & Mischo, C. (2011). Implementing modelling into day-to-day teaching practice – The project STRATUM and its framework. Journal für Mathematik-Didaktik,32(1), 103–131.

Matos, J., & Carreira, S. (1997). The quest for meaning in students' mathematical modelling. In K. Houston, W. Blum, I. Huntley, & N. Neill (ed.) Teaching and learning mathematical modelling (ICTMA 7) (pp. 63–75). Chichester: Horwood.

Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P.L. Galbraith, H.-W. Henn, & M. Niss (ed.), Modelling and applications in mathematics education. The 14th ICMI Study (pp. 3–32). Springer US.

Niss, M., & Blum, W. (2020). The Learning and Teaching of Mathematical Modelling. Milton Park: Routledge.

Plath, J., & Leiß, D. (2018). The impact of linguistic complexity on the solution of mathematical modelling tasks. ZDM Mathematics Education, 50(1–2), 159–171

Pollak, H. O. (1968). On some of the problems of teaching applications of mathematics. Educational Studies in Mathematics, 1(1/2), 24–30.

Pollak, H. O. (1979). The interaction between mathematics and other school subjects. In UNESCO (ed.), New trends in mathematics teaching IV (S. 232–248). Paris: OECD.

Schukajlow, S., Kolter, J., & Blum, W. (2015). Scaffolding mathematical modelling with a solution plan. ZDM - Mathematics Education, 47(7), 1241-1254.

Schukajlow, S., & Leiß, D. (2011). Selbstberichtete Strategienutzung und mathematische Modellierungskompetenz. Journal für Mathematikdidaktik, 32(1), 53-77.

Schukajlow, S., Leiß, D., Pekrun, R., Blum, W., Müller, M., & Messner, R. (2012). Teaching methods for modelling problems and students' task-specific enjoyment, value, interest and self-efficacy expectations. Educational Studies in Mathematics,79(2), 215–237.

Siller, H.-S., & Greefrath, G. (2010). Mathematical Modelling in Class regarding to Technology. Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education, 2136–2145.

Siller, H.-S., Greefrath, G., & Blum, W. (ed.). (2018). Neue Materialien für einen realitätsbezogenen Mathematikunterricht 4: 25 Jahre ISTRON-Gruppe - eine Best-of-Auswahl aus der ISTRON-Schriftenreihe. Springer Fachmedien Wiesbaden.

Stender, P. (2016). Wirkungsvolle Lehrerinterventionsformen bei komplexen Modellierungsaufgaben. Wiesbaden: Springer Spektrum.

Stender, P. (2019): Heuristische Strategien – ein zentrales Instrument beim Betreuen von Schülerinnen und Schülern, die komplexe Modellierungsaufgaben bearbeiten. In I. Grafenhofer, & J. Maaß (ed.), Neue Materialien für einen realitätsbezogenen Mathematikunterricht 6 (pp. 137-150). Wiesbaden: Springer.

Stillman, G. (2011). Applying metacognitive knowledge and strategies in applications and modelling tasks at secondary school. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (ed.), Trends in teaching and learning of mathematical modelling. ICTMA14 (pp. 165–180). Dordrecht: Springer

Stillman, G., Brown, J., & Galbraith, P. (2010). Identifiying challenges within transition phases of mathematical modeling activities at year 9. In R. Lesh, P. Galbraith, C. R. Haines, & A. Hurford (ed.), Modelling students' mathematical modeling competencies ICTMA13 (pp. 385–398). New York: Springer.

Van de Pol, J., Volman, M., & Beishuizen, J. (2010). Scaffolding in teacher-student interaction: A decade of research. Educational Psychology Review, 22, 271–293.

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Vorhölter, K. (2009). Sinn im Mathematikunterricht. Opladen: Budrich.

Vorhölter, K. (2018). Conceptualization and measuring of metacognitive modelling competencies: Empirical verification of theoretical assumption. ZDM – Mathematics Education, 50(1-2), 343-354.

Vorhölter, K., Krüger, A. & Wendt, L. (2019). Metacognition in mathematical modeling – an overview. In S. A. Chamberlin & B. Sriraman (ed.), Affect in Mathematical Modeling (pp. 29–51). Cham: Springer.

Wess, R., Klock, H., Siller, H.-S., & Greefrath, G. (2021 print). Measuring Professional Competence for Teaching Mathematical Modelling: A Test Instrument. New York: Springer.

Zech, F. (1996). Grundkurs Mathematikdidaktik. Weinheim: Beltz.

Zöttl, L., Ufer, S., & Reiss, K. (2011). Assessing Modelling Competencies Using a Multidimensional IRT Approach. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (ed.), Trends in Teaching and Learning of Mathematical Modelling (pp. 427–437). Dordrecht: Springer..