A few teaching principles and their application to fractions

Ron Aharoni
Mathematics, Technion, Israel

Five basic teaching principles

- 1. What before how
- 2. Systematic teaching
- 3. . Concreteness
- 4. Meaning before calculation
- 5. Using precise words.

What before how

Elementary mathematics is deep

Teaching it requires knowing it

Therefore: teach the teachers elementary math, not higher math.

Systematic teaching

math is hard because it is built layer upon layer.

Remedy:

Systematic teaching.

(not missing stages)

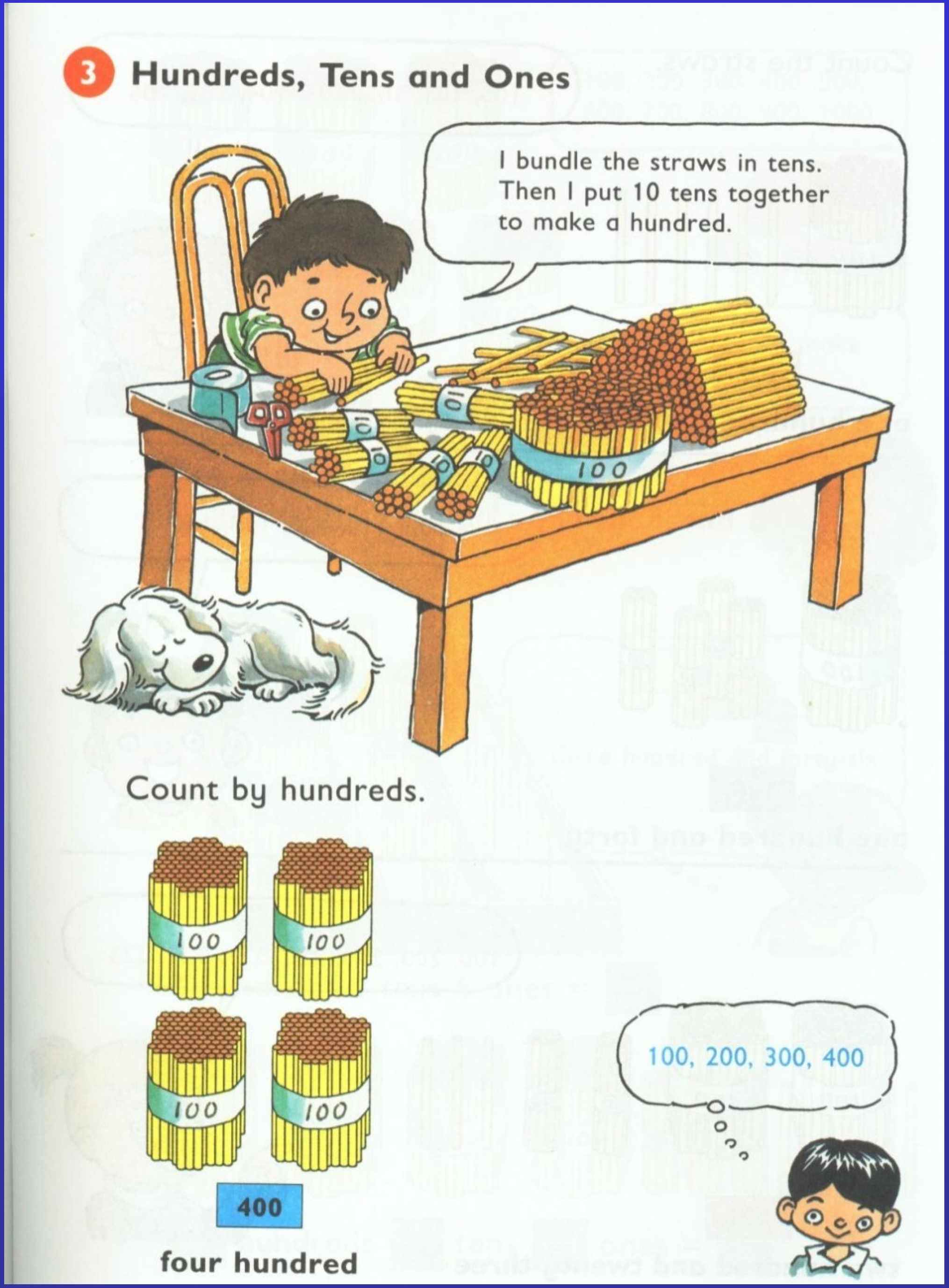
Concreteness

- Example: the teaching of the decimal representation of numbers.

Gathering tens
And hundreds

and thousands

by hand.



Meaning before calcuation

- Example: the meaning of multiplication.

Words

- Examples:
- multiplier and multiplicand
- divisor and devisee
- Sum, difference and quotient

In this talk -

What before how:

The case of fractions

What is a fraction?

- Part of a whole(?)
- Numerator and denominator(?)
- Division (numerator divided by denominator)(?)
- A number that is not an integer(?)

None of these is illuminating

A fraction is a combination of division and multiplication

(In this order)

What is $\frac{5}{8}$ of 240?

Only 25% of Israeli 8-th grade knew the answer in the 1999 international tests.

Nobody ever taught them what is a fraction.

The two steps

a. One eighth of 240 is 240:8, namely 30

b. 5 eighths is what the ear hears: 5 eights. Namely 5 times 30.

The fraction is the combination of two operations

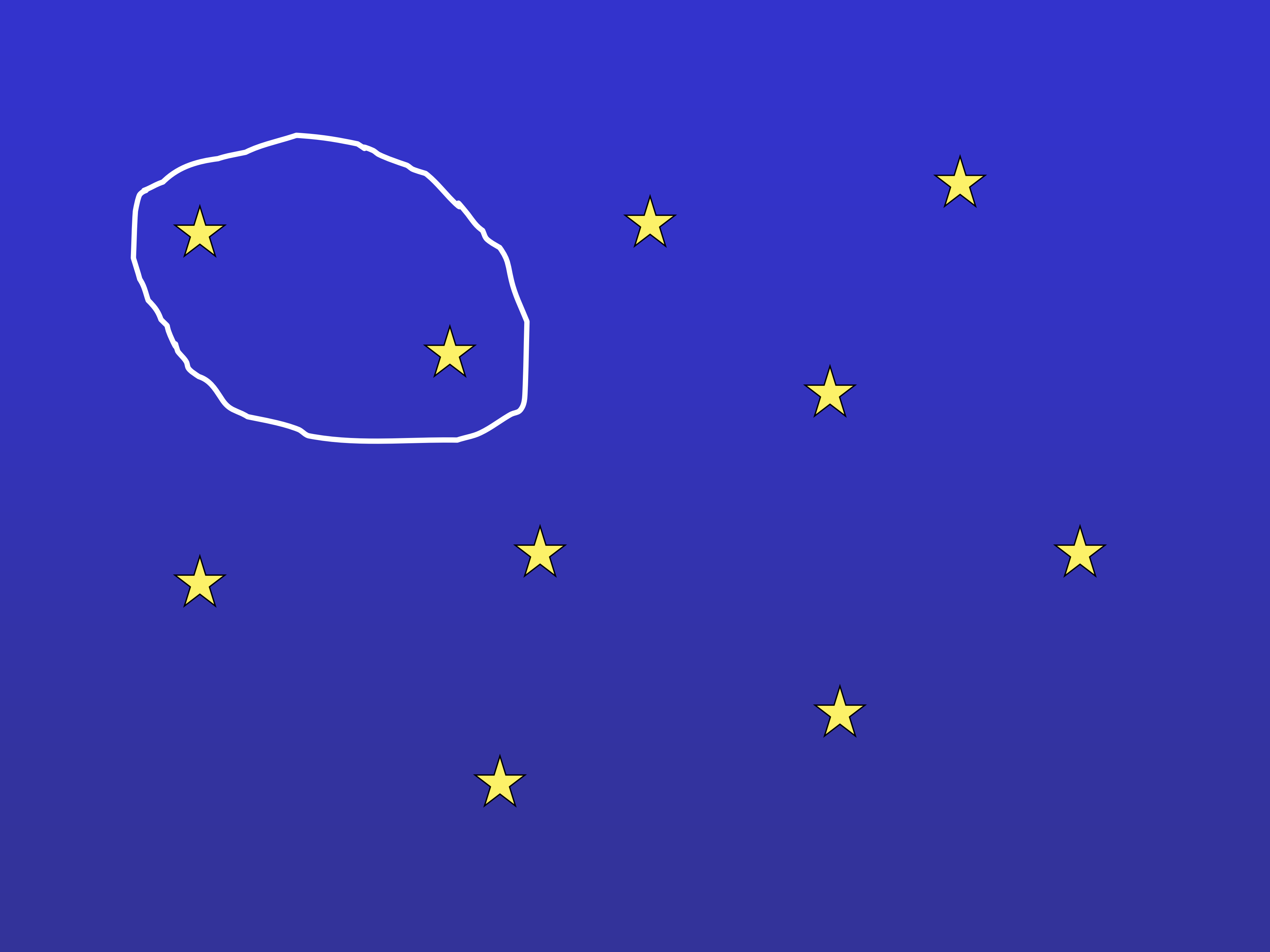
Taking $\frac{5}{8}$ of something means:

a. Dividing by 8

b. Multiplying the result by 5.

A remedial lesson on fractions

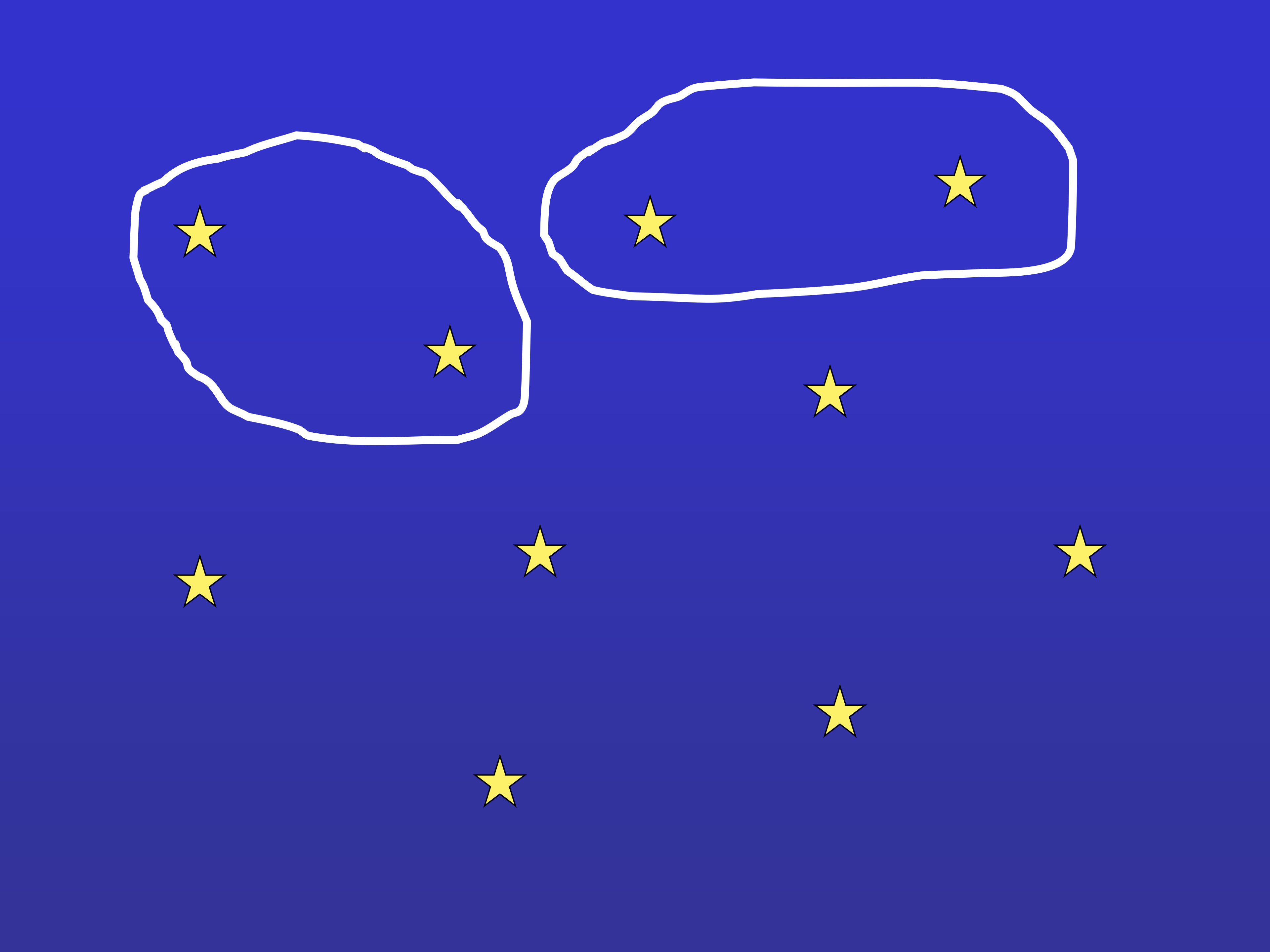
Circle one fifth of the stars



One fifth of 10 is 2.

$\frac{1}{5}$ of 10 is 2.

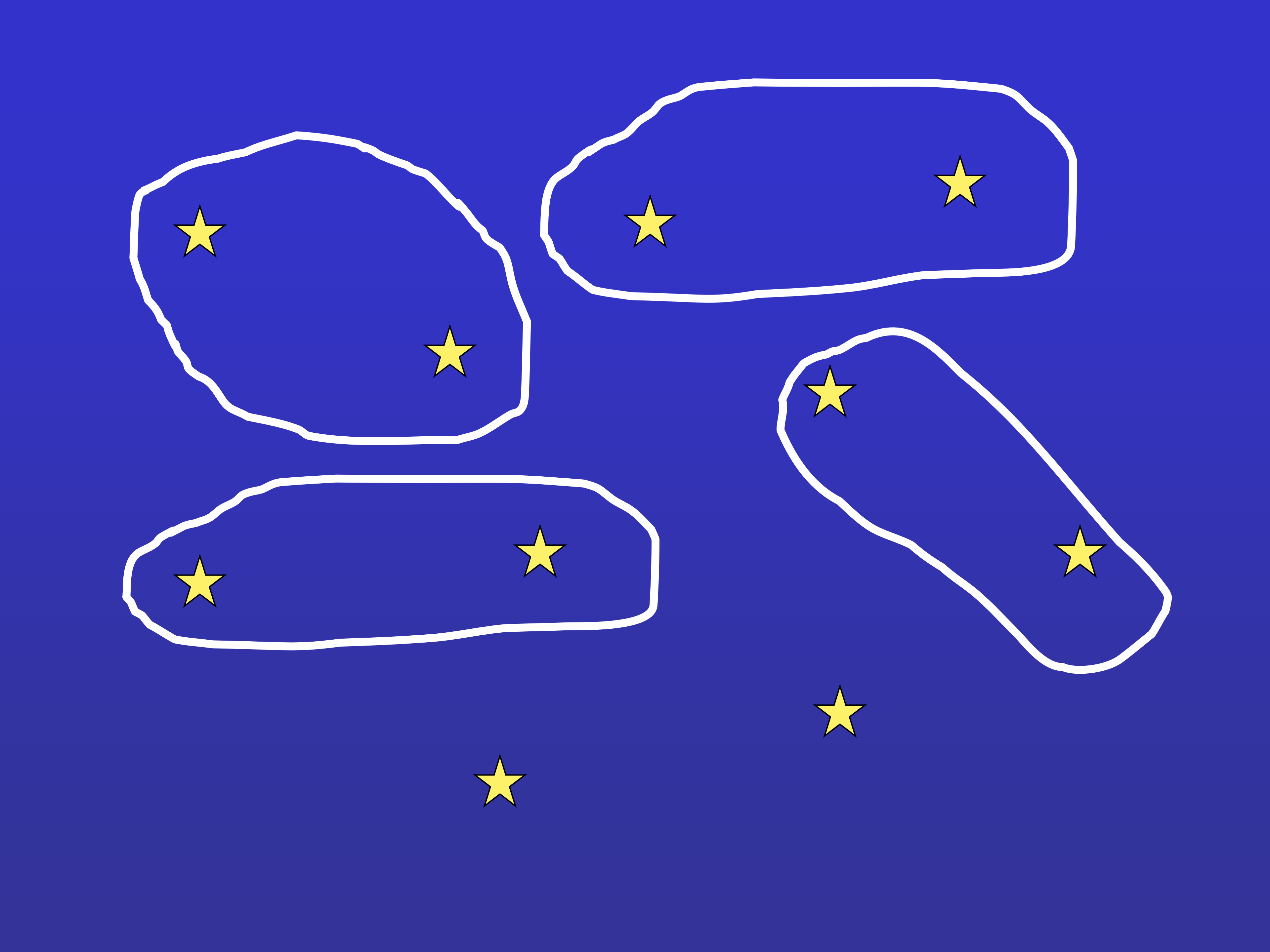
Now circle another fifth

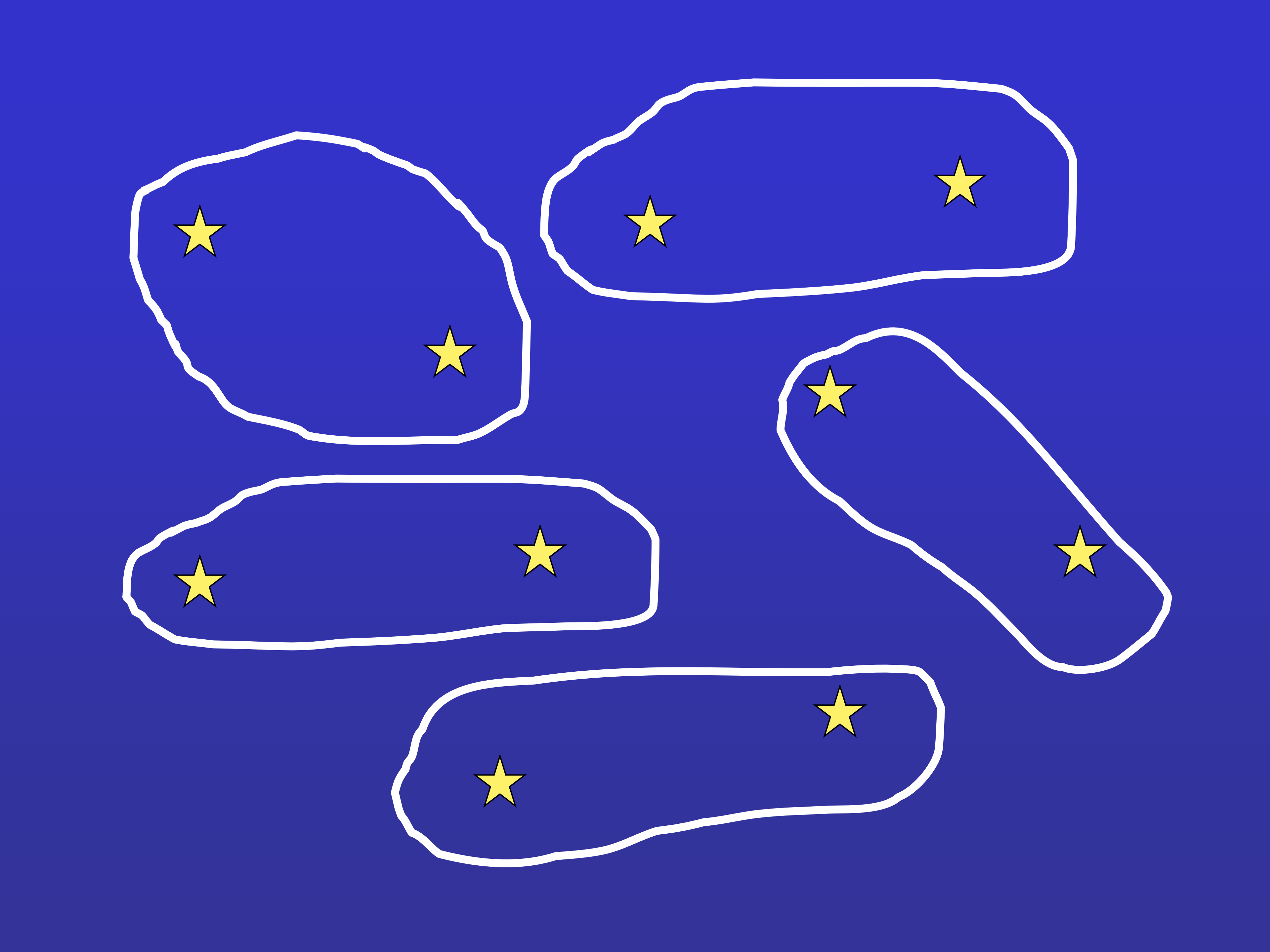


Two fifths of 10 are 2 times 2, which is 4.

$\frac{2}{5}$ of 10 are 4.

circle another fifth





$\frac{4}{5}$ of 10 is 8.

$\frac{5}{5}$ of 10 is 10.

A teaching principle – repeat until they laugh

What are 5 fifths of 10?

What are 5 fifths of 20?

What are 5 fifths of 100?

What are 5 fifths of a million?

When the kids laugh, it means they understood.

Now – to imaginary fractions:

What are 6 fifths of 10?

Can you draw it?

What are 100 fifths of 10?

The starting point - division

Egyptian fractions – numerator 1

An Egyptian fraction is nothing but division

Taking $\frac{1}{5}$ is plainly dividing by 5.

Namely, dividing into 5 equal parts, and taking one of them.

Fractions should be taught together with division

- When you learn to divide by 2, define “half”, including the notation. Yes, even in Grade 1.

- In Grade 2 you teach $\frac{1}{3}$

- And then $\frac{2}{3}$: two thirds is just “two thirds”.
- Two thirds of 6 apples is 2 times 2 apples.

The (wrong) separation of fractions and division

a. Timing: division is taught in Grade 2, fractions in Grades 3, and mainly 4.
b. What is divided: fractions are taken of shapes, division is of numbers.

The reason for the separation:

Two false premises

a. “Division is the opposite of multiplication, and multiplication is of numbers”
b. A fraction should be taken of something that looks like a whole, and numbers are not wholes.

The first assumption is wrong

- Multiplication is not only of numbers
- You can multiply an apple by 2.
- Two times an apple is just 2 apples.

An untold secret

- Multiplication and counting are the same thing.

And you can also divide an apple

Shapes and body can and should be divided

Dividing an apple and a rectangle by 3 should precede dividing 6 by 3.

The second premise is also wrong:

- A whole does not have to look like one block.
- A set can also be a whole.
- In fact, the children have met it before.

The first arithmetic operation: forming a whole

A set can be a whole

First example: the decimal system

It is based on taking ten objects and declaring them to be one object – a “ten”

Then these objects can be counted, and be gathered to tens – ten tens are a called “a hundred”.

Second example of a set as a whole:

multiplication

- 3 times 4 means taking 4 elements and considering them as one unit
- And then repeating it 3 times.
- (yet again): Multiplication is counting.

Two mysterious questions:

A. Why is $\frac{2}{3}$ of 24 the same as $\frac{2}{3} \times 24$?

Because multiplying is counting

Two apples = 2 times apple = 2 x apple

Similarly, $\frac{2}{3}$ of an apple = $\frac{2}{3}$ times an apple

= $\frac{2}{3}$ x apple.

Similarly,

$\frac{2}{3}$ of 24 apples =

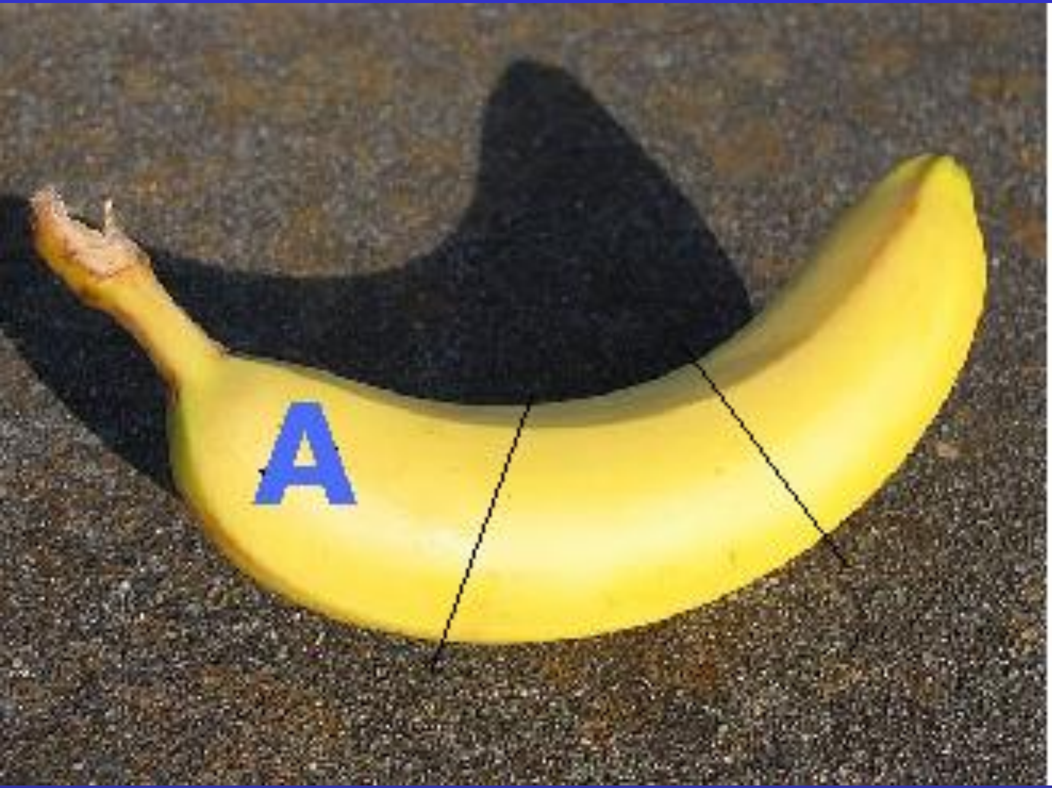
$\frac{2}{3}$ times 24 apples =

$\frac{2}{3}$ x 24 apples.

B. Why is $\frac{2}{3}$ the same as 2:3?

How do you divide an orange and a banana among 3 people, equally?

Of course, divide each into 3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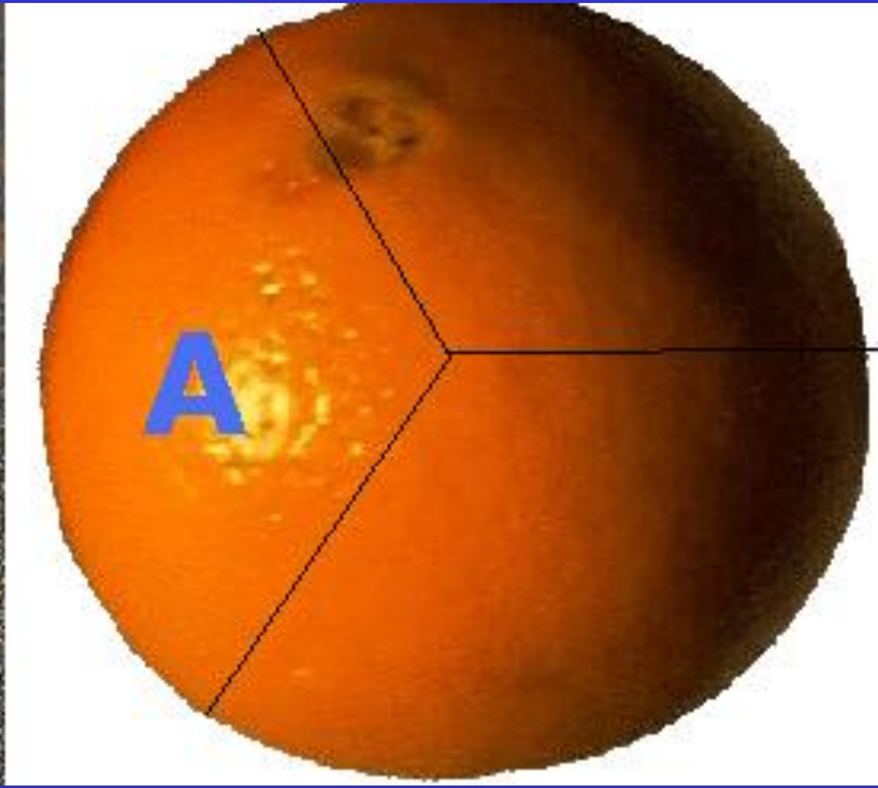
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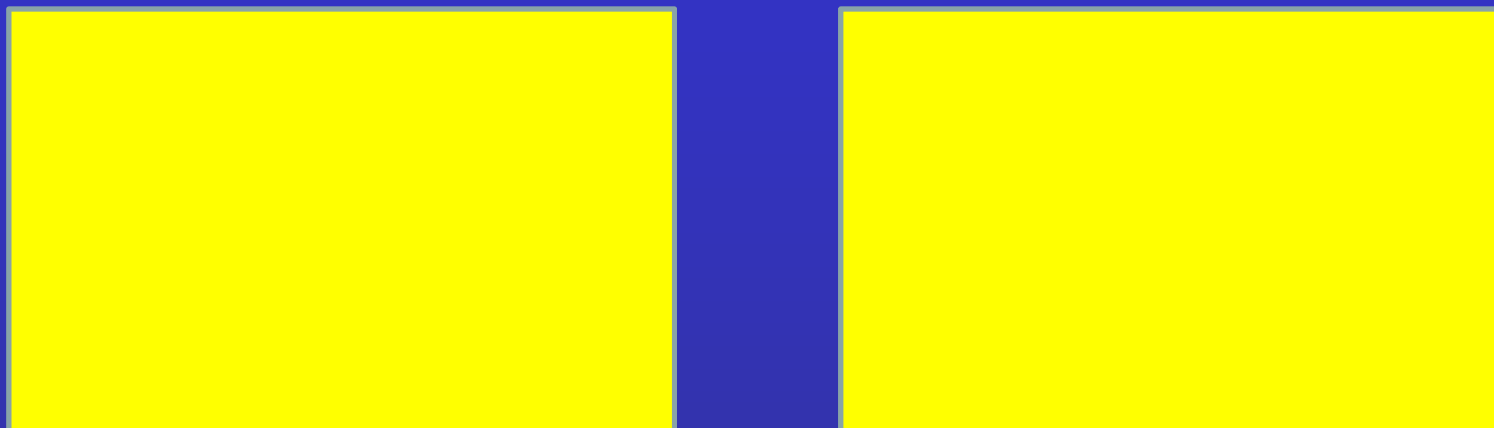
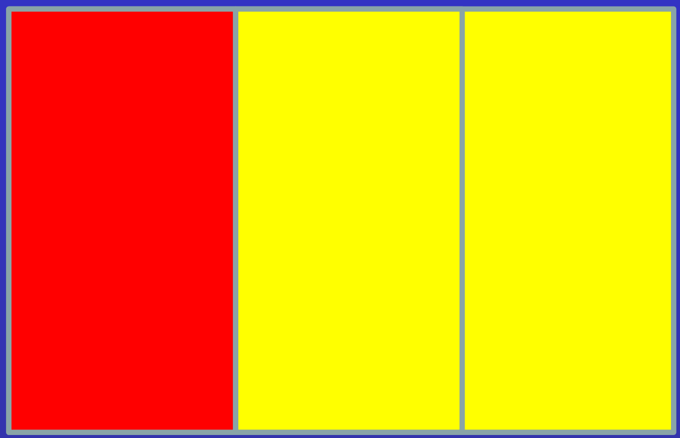
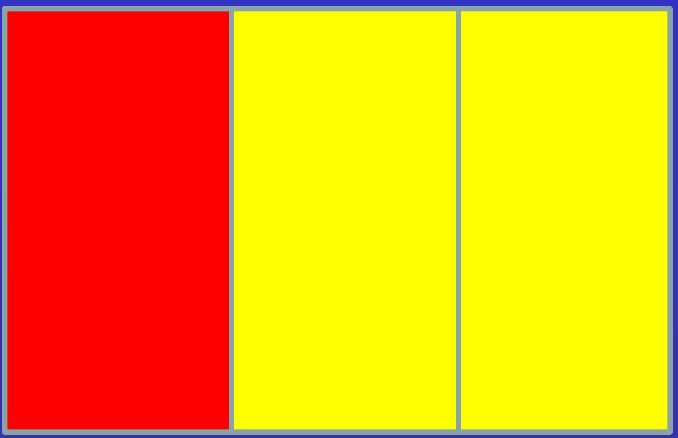
How do you divide 2 rectangles into 3?



So, 2 rectangles divided by 3 is 2 thirds of a rectangle

$$\mathbf{2:3} = \frac{2}{3}$$

Tres “trucos” para la enseñanza

. Nothing is too simple

Always give the simplest example.

There is no such thing as “too simple”.

Even more importantly – encourage the students to give the simplest example

Ask the students what is the simplest case of division

(of course, dividing by 1)

What is the simplest fraction?

Not a half, but $\frac{1}{1}$

Trick no. 2: repeat until they laugh

- What is a half of 2 apples?
- And a $\frac{1}{3}$ of 3 apples?
- And $\frac{1}{4}$ of 4 apples?
- When they laugh, they have understood.

Trick no 3: ask the students to provide examples themselves

In fact, this is more than a trick –
it is a basic principle.

For example:

- Give and example of a fraction smaller than 1/100.
- An example for an equation with solution 5.
- An example for two fractions whose sum is 1.
- An example for two fractions whose product is 1
- An example for two fractions whose product is 2
- An example for two fractions whose product is 3
- (repeat until they laugh!)
- An example of a fraction between ½ and 1.

Part II: operations on fractions

Where to start

(The answer is not at all obvious)

Not with addition and subtraction, but with multiplication

- Reason 1: fractions are born from multiplication and division, and hence they behave better with respect to these operations
- Reason 2: multiplication and division are necessary for the idea of the common divisor.

Two organizing principles

Multiplying the numerator by 3 multiplies the fraction by 3.

Multiplying the denominator by 3 divides the fraction by 3

Dividing twice

- What happens if we divide an apple to 2 parts, and afterwards each part to 3:
- We have 2×3 parts, so in fact we divided by 6
- Similarly, dividing by 4 and then by 5 is dividing by 20.

In fractions language:

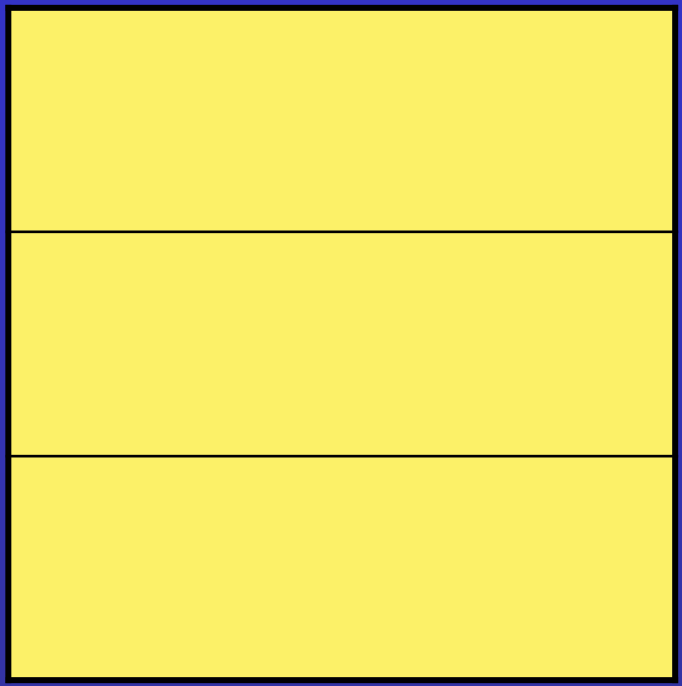
$$\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$$

$$\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$$

When the denominator is multiplied by 4, the fraction is divided by 4

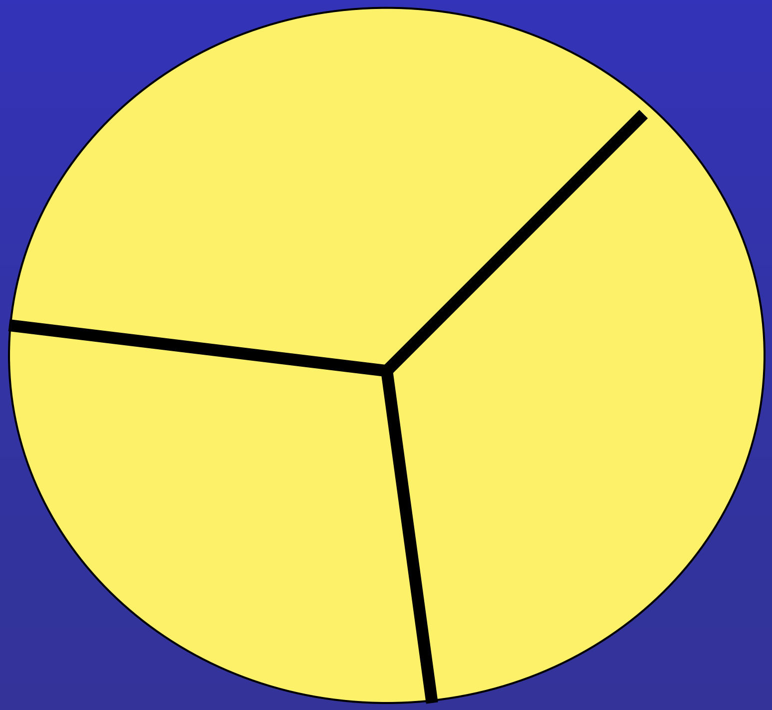
A didactic comment: don’t use magic

dividing by 3, and then by 4 is easy to do using a rectangle:

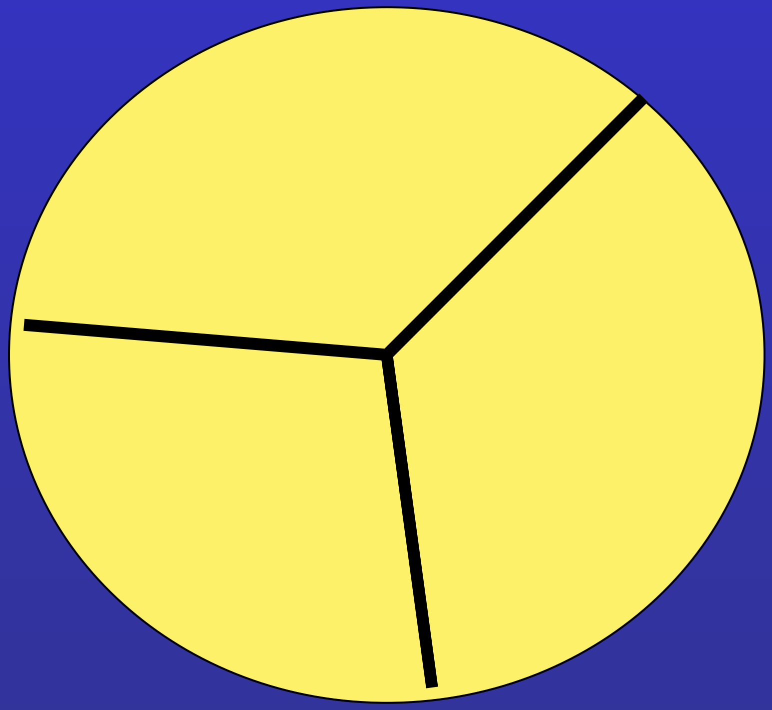


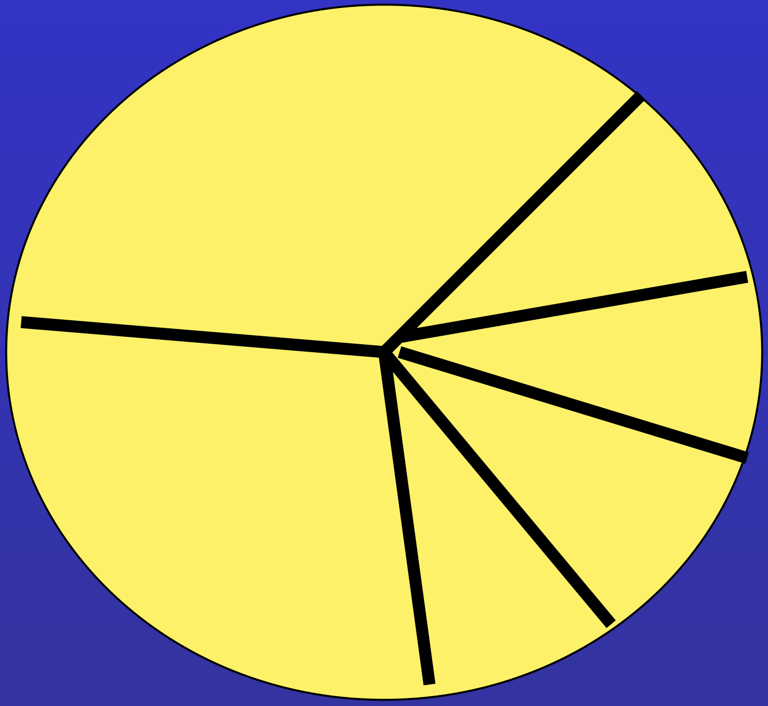
But this looks like magic.

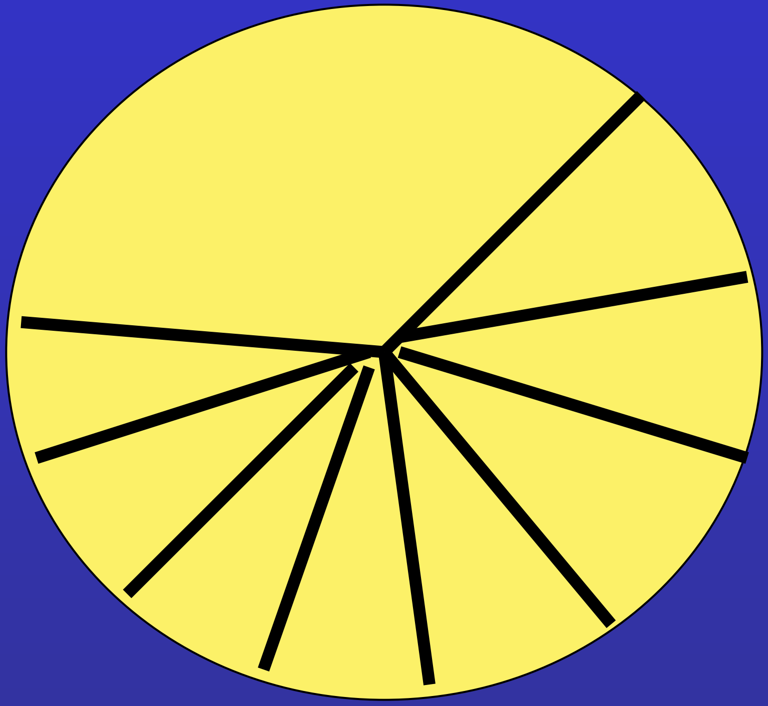
Let the children do it themselves, say with a pizza:

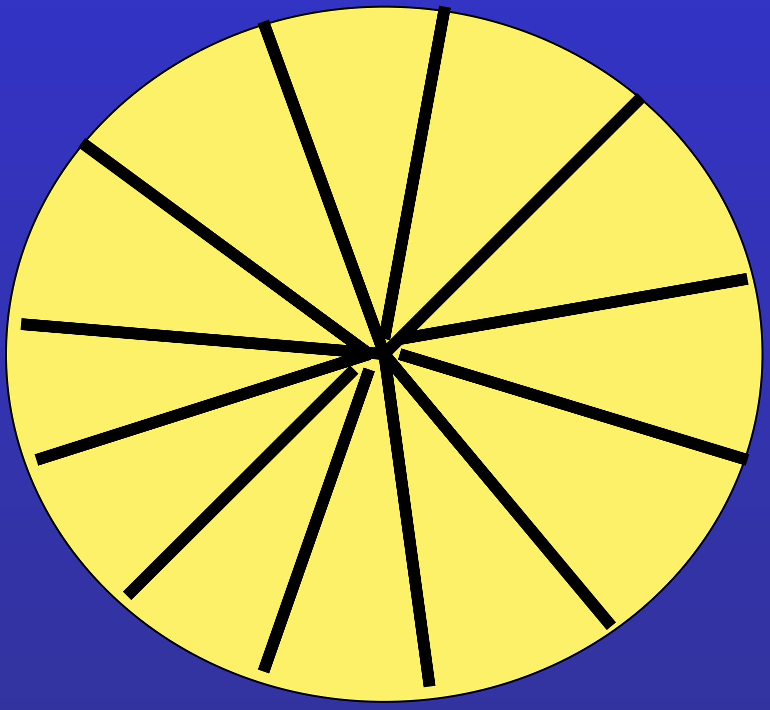


Ask them to divide each part into 4:

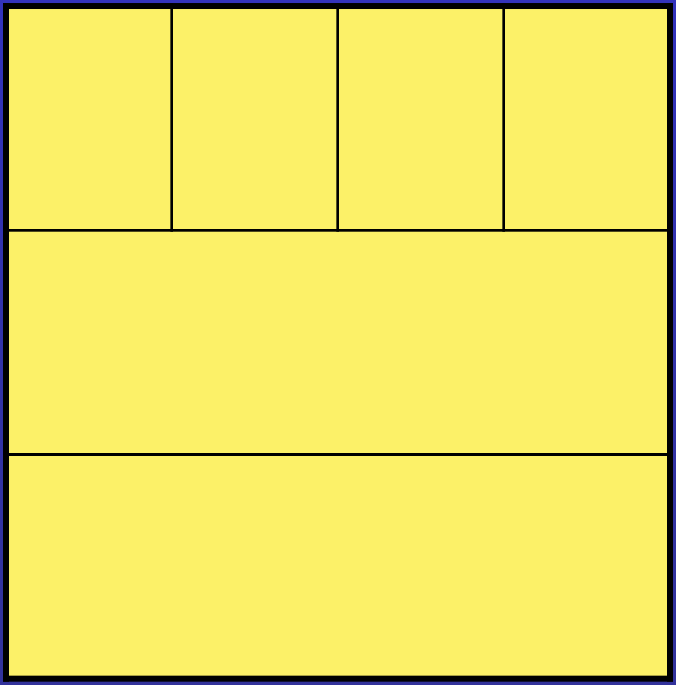


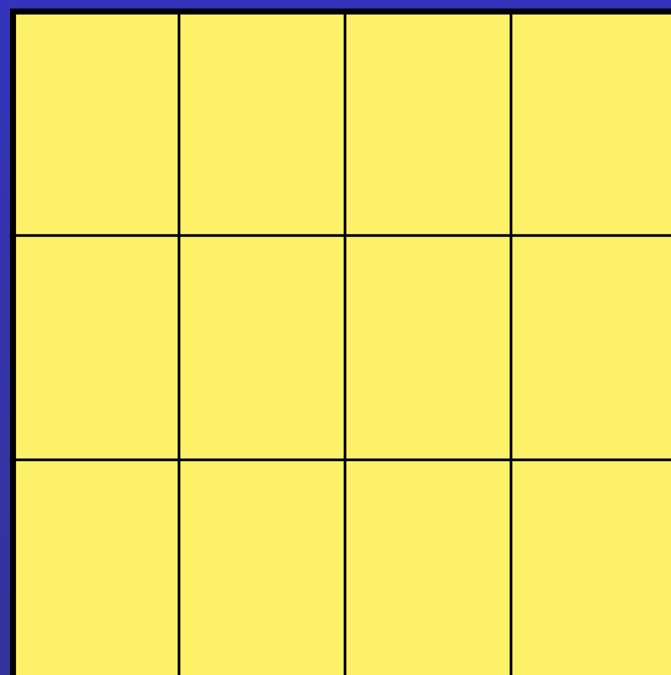






Or with rectangles, but step by step:





Another example: what is $\frac{1}{3} \times \frac{1}{4}$

- Multiplying $\frac{1}{4}$ by a third is taking a third of $\frac{1}{4}$.
- Which is dividing $\frac{1}{4}$ by 3.
- Which means dividing by 4, and then by 3, which is dividing by 12.

So,

$$\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$

What happens when you multiply the numerator by 3?

Ask: what is more – 6 apples, or 2 apples?

How many times more?

And now:

What is more, $\frac{2}{7}$ or $\frac{6}{7}$?

How many times more?

A teaching trick: ask half a question

Don’t ask how many times is $\frac{6}{7}$ larger than $\frac{2}{7}$.

Ask just: which is larger?

The students will ask themselves – how many times larger?

Conclusion:

When the numerator grows 3 times, the fraction grows 3 times

Expansion

What happens if you first multiply the numerator by 4,
and then the denominator by 4?

The number grows 4 times, and then gets smaller 4 times

So it returns to be the same.

How to multiply fractions

First fact:

$$\frac{2}{3} = 2 \times \frac{1}{3}$$

This is nothing but the definition of “two thirds”

$$\frac{2}{3} = 2 \times \frac{1}{3}$$ And hence -

Multiplying by $\frac{2}{3}$ means multiplying by 2 and dividing by 3.

And this, we know, means multiplying the numerator by 2 and the denominator by 3.

$$\frac{2}{3} \times \frac{7}{5} = \frac{2 \times 7}{3 \times 5}$$

Division of fractions

What is $10 : \frac{2}{3}$?

"Ask me a simpler question" –

what is the simplest division?

- Of course, 1:1. Or 10:1.
- What is the simplest division by a real fraction? - $1:\frac{1}{2}$

- How many times does ½ go into 1?
- What about $10 : \frac{1}{2}$?
- If ½ goes 2 times into 1, it goes 10 x 2 into 10.

- What is $3 : \frac{1}{2}$? 6
- And $5 : \frac{1}{2}$? 10
- What is the rule?
- Dividing by ½ is multiplying by 2.

- Returning to the original question – what is

$$10 : \frac{2}{3}$$

- $10 : \frac{1}{3} = 30$

- What is more, $10 : \frac{1}{3}$ or $10 : \frac{2}{3}$?

- If $\frac{1}{3}$ goes 30 times into 10, how many times does $\frac{2}{3}$ go into 10?

More, or less than 10?

How many times less?

Common denominators

The pizza man and the indecisive customer

A customer in a pizza shop is not sure:

Is he going to have to divide the pizza between 2 people, or 3.

What number of parts will work in both cases?

Of course, 6.

- And if there may be either 2 or 4 parts?
- How about 4 and 5?
- And 4 and 6?

What is the simplest?

- If he vacillates between 2 and 2.
- Or even between 1 and 1.
- The next simple case – between 1 and 2.
- Encourage the students to find the simplest cases, in order.